

IC/95/15
hep-ph/9502241

An Introduction to the Heavy Quark Effective Theory

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Abstract

These lecture notes begin with a brief survey of the physics of heavy quark systems. This discussion motivates the introduction of the Heavy Quark Effective theory (HQET) which captures a great deal of the intuition developed. A derivation of the HQET from QCD is presented as well as an analysis of its special properties. The effective theory can be seen to amount to a one-dimensional field theory in the quark sector. The heavy quark flavour- and spin- symmetry of the effective lagrangian is an offspring of this. Other topics covered include the question of covariance of the theory, the construction of interpolating fields for the heavy hadron states, the application of LSZ reduction theorems to determine the (reduced) number of form factors in flavour changing transitions, a complete verification of Luke's theorem, plus the matching conditions between QCD and the HQET beyond tree level.

These are an expanded version of lectures presented during the Trieste 1994 Summer school on High Energy physics.

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1 Introduction

QCD as the theory of strong interactions has been with us for over twenty years. It has been remarkably successful in describing high energy physics. The discovery of asymptotic freedom has allowed for many perturbative calculations of physical quantities within QCD when combined with the parton model, such as Drell-Yan processes, deep inelastic scattering and various scaling relations just to name a few. These successes notwithstanding QCD has proved to be intractable in the infrared regime by direct analytic means. For example we do not have a handle on confinement. One, at this point, normally needs to employ lattice gauge theory or to pass to QCD inspired effective theories such as non-linear sigma models or potential models of various kinds. Put rather blankly-QCD in this regime is **hard**.

However, during the last few years there has been a resurging interest in heavy quark physics within the context of QCD. The reason for this is that one has been able to extract general principles from particular models of heavy flavour transitions. In the work of [1] one can see the importance played by the velocity of the heavy quarks (as opposed to their momentum); indeed hidden in the analysis is a peaking about a velocity super selection rule [2]. It is to the credit of Isgur and Wise [3] that they were able to extract the physical, model independent implication, namely, that for infinitely heavy quarks, velocity is the only important parameter.

Further,

- (i) there is a heavy quark flavour symmetry,
- (ii) there is a two-fold spin degeneracy (because the spin coupling is $\propto 1/m_Q$, which tends to zero), and
- (iii) at the zero recoil point, or equivalently at maximum momentum transfer, the elastic transition is absolutely normalized (following Voloshin and Shifman [4]).

Our understanding of the dynamics has not improved. However, the new symmetries in heavy quark theory give rise to numerous predictions for free!

The heavy quark effective theory HQET, as it has come to be known, captures the physics of the heavy quark systems which brings to light these new symmetries. HQET, as we will see, is an expansion of the QCD action in inverse powers of the heavy quark mass. Heavy quarks Q , for present purposes will be (c, b, t) with masses $m_Q \approx (1.5, 5, 175)$ Gev. The scale is set by $\Lambda_{QCD} \simeq 330$ Mev and the expansion parameter is typically Λ_{QCD}/m_Q . See Table 1. While the top quark seems to be a theorists delight, the expansion parameter being about 10^{-2} , it unfortunately decays far too fast to be useful.

Working within the context of lattice theory Eichten and Hill [5] had suggested

| spin 0^- | spin 1^- |
|--------------------------------------|--|
| $m_D = 1.86$ Gev | $m_{D^*} = 2.01$ Gev |
| $D^0 = c\bar{u}$ or $D^+ = c\bar{d}$ | $D^{*0} = c\bar{u}$ or $D^{*+} = c\bar{d}$ |
| $m_B = 5.28$ Gev | $m_{B^*} = 5.32$ Gev |
| $B^- = b\bar{u}$ or $B^0 = b\bar{d}$ | $B^{*-} = b\bar{u}$ or $B^{*0} = b\bar{d}$ |

Table 1: Charm and bottom mesons.

a non-covariant effective action for the infinitely heavy quark limit of QCD, based on previous work of the related, but somewhat different, non-relativistic limit of QCD by Feinberg [6] and by Caswell and Lepage [7]. This action was covariantised by Georgi [8] and subsequently a great deal of effort has been expended on determining the implications of this theory, both on the lattice and in the continuum, [9]-[30]. One of the important developments has been the observation by Luke [21] that in the case of heavy meson transitions the Voloshin-Shifman normalization at zero recoil remains valid even when $1/m_Q$ corrections to the lowest order effective theory are taken into account. This no-correction theorem has been generalised and proven in various ways in the literature [17], [22].

A derivation of the effective theory which also gives a systematic expansion to any order in $1/m_Q$ was given in [23], [24] and [13]. Previously, corrections to the infinitely heavy quark effective theory had been obtained by matching the results to QCD. As in [23] the heavy quark effective theory is obtained from QCD directly by a series of Foldy-Wouthuysen type of field redefinitions, the matching, at tree level is immediate. The same is true for the derivation given in [8]. That one needs to ‘improve’ the relationship between QCD and the HQET at one and higher loop level comes about because regularization does not respect the derivations.

These notes are based on an unpublished manuscript written with J. G. Körner and S. Balk and on a talk delivered at the 1992 DESY workshop given by the second author. As the reader will see, we have also benefited from the review articles of M. Neubert [29] and B. Grinstein [30]. The notes have turned out rather long on account of the fact that we attempt to give a rather complete account of the properties of the heavy quark effective theory at $\mathcal{O}(1)$ and $\mathcal{O}(1/m_Q)$. It has been our intention to give a critical account of the assumptions that go into the formulation of the HQET. We have tried to be clear on what is derived and what is intuited, pinpointing the assumptions that are being fed into the analysis. Our first aim then is to make precise what we can, with confidence, say about heavy quark physics. The following picture emerges.

1.1 Physics of a Heavy Hadron

The classical picture of a heavy hadron that one has is in many ways similar to that of the Hydrogen atom. The typical momentum carried by the light degrees

of freedom (light quarks plus glue) inside a hadron Λ_{QCD} is of the order of the proton mass divided by three $\sim 330\text{Mev}$. This is a measurement of how far the light quarks are “off” shell or, rather more correctly, this is about what their ‘constituent’ masses are. Typically, then, the light quarks and their gluonic cloud are carrying momentum Λ_{QCD} . In heavy hadrons the ‘mass’ of the heavy quark m_Q is much greater than the typical scale, $m_Q \gg \Lambda_{QCD}$, and the heavy quark is carrying most of the momentum of the heavy hadron. The interactions of the heavy quark with the light degrees of freedom will also only change the momentum of the heavy quark by the order of Λ_{QCD} , so that the heavy quark is then almost on mass shell. Indeed one expects $M_Q \approx m_Q + 0(\Lambda_{QCD})$.

While the change in momentum of the heavy quark is of order Λ_{QCD} , its change in velocity, $\Lambda_{QCD}/m_Q \ll 1$, is negligible as the mass of the heavy quark goes to infinity. The picture one has then is of the heavy quark moving with constant velocity and this velocity is that of the heavy hadron. It is important to emphasise that it is *not* momentum that is being equated but, rather, velocity. In the rest frame of the heavy hadron the heavy quark is almost at rest; it is only slightly recoiling from the emission and absorption of soft gluons. This picture does not depend on the actual value of m_Q but just that it satisfies $m_Q \gg \Lambda_{QCD}$. As the mass of the heavy quark is taken to be bigger and bigger the recoil is less and less until ultimately, in the limit $m_Q \rightarrow \infty$, the heavy quark does not recoil at all from the emission and absorption of soft gluons. In this limit the heavy quark acts, therefore, like a static colour source. It is clear that in this limit the binding is independent of the flavour and hence the difference between the mass of the heavy hadron and the heavy quark, $\bar{\Lambda} = M_Q - m_Q$, is a universal, flavour independent, constant.

In many ways we have also just described the Hydrogen atom. Take the proton to be a fundamental particle. The typical momentum imparted to the proton by the electron and the photon cloud is very much smaller than the mass of the proton (m_p). The proton can be taken to be a static photon source which binds the electron to form the Hydrogen atom. The fact that the Schrödinger equation for the electron in a $1/r$ potential describes quantitatively the Hydrogen atom so well is indicative of the success of this picture. One of the things that is missed by this ‘non-relativistic’ analysis is the hyperfine splitting of energy levels. But such corrections are rather small compared to the energy level, $\Delta E/E \ll 1$. One can derive the Schrödinger equation from the fully relativistic and interacting Dirac theory for the Hydrogen atom and systematically incorporate corrections such as the Thomas term for the spin-orbit coupling.

Likewise for the heavy hadron, it is immaterial, in a first approximation, what the spin state of the heavy quark is and, in analogy to the above discussion, one can give a systematic derivation of corrections to this picture. The corrections will be of the form of a series in $1/m_Q$ with the spin coupling coming in at next to leading order. One of our aims in these notes is to give a derivation from QCD of this ‘effective’ theory whose lowest term is analogous to the static proton action

in QED.

We do not need, however, to get into any highly abstract formal considerations in order to exhibit the features mentioned above within the context of QCD [19]. Let us simply take the $m_Q \rightarrow \infty$ limit of the lowest order propagator and connected three point function for the heavy quark near its ‘mass-shell’, that is with $p_Q = m_Q v + k$, where v is the velocity of the heavy hadron and the components of k are bounded by Λ_{QCD} . The first of these behaves as,

$$\frac{i}{\not{p}_Q - m_Q} = i \frac{(1 + \not{\gamma})}{2v \cdot k} + O(k/m_Q). \quad (1)$$

This asymptotic form already exhibits some of the features of the physical arguments. Recall that QCD with massless quarks has a flavour symmetry. This symmetry is (softly) broken by the introduction of different masses for the different flavours. Clearly there is no dependence on the mass at lowest order in (1) and so, just as for massless quarks, there is no dependence, at this order, on the flavour of the quark.

The second important feature is that the lowest order term is diagonal in spin. The $(1 + \not{\gamma})$ projector picks out the particle state in the field while $(1 - \not{\gamma})$ projects out the anti-quark component of the field. The rest of the propagator is proportional to the unit matrix and so does not depend on the spin of either the quark or anti-quark components.

The three point Greens function (we amputate the gluon leg)

$$\frac{i}{\not{p}_Q - m_Q} (ig\gamma^\nu) \frac{i}{\not{p}_Q + \not{q} - m_Q} \quad (2)$$

with soft gluon momentum q (also bounded by Λ_{QCD}) goes like

$$igv^\nu \frac{(1 + \not{\gamma})}{2} \frac{i}{v \cdot k} \frac{i}{v \cdot (k + q)} + O(\Lambda_{QCD}/m_Q). \quad (3)$$

This expression also picks out the quark component and does not depend on spin to leading order.

To summarise: the heavy quark symmetry is the statement that

$$\mathbf{HEAVY\ HADRON}_1(\mathbf{spin}_1) \equiv \mathbf{HEAVY\ HADRON}_2(\mathbf{spin}_2)$$

1.2 Analogies and Differences

We can get some more information as regards the physics of heavy quark theory by pushing the analogy with the Hydrogen atom further. At some point the analogy must breakdown and we will have to come to grips with that as well.

Recall that the Schrödinger equation makes no reference at all to the nucleon and so it *is* invariant under the symmetry described above. For the moment let us think of the different isotopes of Hydrogen (p+e), Deuterium (pn+e) and Tritium (pnn+e), as arising from different ‘flavours’ of the proton. As far as the electron is concerned the photonic field is the same regardless of the flavour. Treating the nuclei as infinitely massive the Schrödinger equation describes these systems well. The spectra of the three (ignoring reduced mass corrections) are identical. This coincides with the heavy quark symmetry which allows one to exchange heavy quarks without effecting the spectra. Of course the Schrödinger equation is blind to the spin of the electron relative to the proton. At this order of approximation this is the other part of the heavy quark symmetry, namely that the spin of the heavy quark is immaterial.

What the Schrödinger equation misses is the hyperfine splitting of the energy levels. A more correct account follows from an analysis of the Dirac equation. Still treating the nuclei as very heavy the predictions of the Dirac equation differ from those of the Schrödinger equation at order $1/m_p$. The correction of interest is a Pauli-term with the typical $\bar{\psi}\sigma^{\mu\nu}F_{\mu\nu}\psi$ coupling. The corrections can be systematically derived by a series of Foldy-Wouthuysen transformations. We will apply these to QCD.

Now it is time to mention the major difference between QCD and QED. Field theoretically we know that $\alpha_e (e^2/4\pi)$ and $\alpha_g (g^2/4\pi)$ run. For large distances α_e tends to zero while α_g grows. The reason that one can perform with some reliability the calculation of the spectra of the Hydrogen atoms is that the coupling constant is small. Now the typical distance is the Bohr radius $a = h^2/(2\pi e)^2 m_e \approx .5 \times 10^{-10} \text{ cm} = 5 \times 10^4 \text{ fermi}$ so the binding occurs at “large” distances compared to the typical length scales of the nucleons which are of the order of fermi’s. However, in QCD although, again, binding is a long distance phenomena, the coupling constant is large and perturbation theory is therefore not reliable.

Recall that, in the static proton approximation, the potential for an electron in the photonic field of the proton depends on distance as $1/r$. One obtains this behaviour in the Born approximation by consideration of figure 1.

The corresponding amplitude is

$$\bar{u}_P(p+k)\gamma_\mu u_P(p)D^{\mu\nu}(k)\bar{u}_e(q-k)\gamma_\nu u_e(q) \quad (4)$$

where $p' - p = k$, and $p = m_P v$ and $v = (1, \vec{0})$. Apply now the Gordon decomposition,

$$\bar{u}_P(p+k)\gamma^\mu u_P(p) = \bar{u}_P(p+k) \left(\frac{(2p+k)^\mu}{2m_P} + i \frac{\sigma^{\mu\nu}k_\nu}{2m_P} \right) u_P(p) \quad (5)$$

on the proton side to arrive at

$$\bar{u}_P(p+k) \left(\frac{(2p+k)^\mu}{2m_P} + i \frac{\sigma^{\mu\nu}k_\nu}{2m_P} \right) u_P(p) D_{\mu\sigma}(k) \bar{u}_e(q-k) \gamma^\sigma u_e(q). \quad (6)$$

In a covariant gauge the photon propagator is

$$D^{\mu\nu}(k) = \frac{(\eta^{\mu\nu} - k^\mu k^\nu/k^2)}{k^2}. \quad (7)$$

Substituting this into (4), and remembering that $\bar{u}(p+k) \not k u(p) = 0$, one obtains

$$\bar{u}_P(p+k) \gamma_\mu u_P(p) \frac{\eta^{\mu\nu}}{k^2} \bar{u}_e(q-k) \gamma_\nu u_e(q). \quad (8)$$

Now, as $p^2 = p'^2 = m_P^2$, we find that $2m_P k_0 + k_0^2 - \vec{k}^2 = 0$. In the infinite mass limit, so as not to have an infinite momentum transfer, one finds that $k_0 \approx \vec{k}^2/2m_P$. Notice that in the heavy proton limit, $k/m_P \rightarrow 0$, the Gordon decomposition goes to $\bar{u}_P(p+k) v^\mu u_P(p)$, which is consistent with our tree level analysis of the vertex in that limit. As $k_0 \rightarrow 0$, (6) reduces to

$$\frac{2m_P}{-\vec{k}^2} \bar{u}_e(q-k) \gamma^0 u_e(q) + O(1/m_P). \quad (9)$$

This reproduces the usual result of the coulomb scattering of an electron in a given external electromagnetic field (generated by a heavy nucleus) $A_\mu(x) = (ie/4\pi r, \vec{0})$.

It is only the 0-component of the gauge field which enters finally so that one is tempted, as we will be tempted in due course, to ask what happens in the gauge $A_0 = 0$? Physics is gauge invariant, so that going to this gauge changes nothing. Indeed the photon propagator in the gauge $v.A = 0$ is

$$D_{\mu\nu}(k, v) = (\eta_{\mu\nu} - \frac{v_\mu k_\nu + v_\nu k_\mu}{v.k} + \frac{k_\mu k_\nu}{(v.k)^2}) \cdot \frac{1}{k^2} \quad (10)$$

and when sandwiched between the currents only the $\eta_{\mu\nu}$ part survives. The calculation is the same as in the covariant gauge and one reproduces (9).

A problem arises when one passes to the heavy proton limit first. The naive limit allows us to replace (5) with

$$\bar{u}_P(p+k) v^\mu u_P(p) \quad (11)$$

but, in the gauge $v.A = 0$, the photon propagator satisfies $v^\mu D_{\mu\nu}(k, v) = 0$, so that at first sight, it seems that there will be no scattering at all! The resolution of this puzzle is that in non-covariant gauges spurious poles in the propagator mix terms of different orders in $1/m_P$. Let us first dispense with the term $\bar{u}_P(p+k) \sigma_{\mu\nu} k^\nu u_P(p) D^{\mu\sigma}(k, v) \bar{u}_e(q-k) \gamma_\sigma u_e(q)$ in (6). By current conservation and anti-symmetry of $\sigma_{\mu\nu}$ the only part of the propagator that survives is that proportional to $\eta_{\mu\nu}$. This is the same as for the covariant gauges and can safely be ignored in the limit. In passing to the limit (11) we have dropped a term

$$\bar{u}_P(p+k) \frac{k_\mu}{2m_P} u_P(p) \quad (12)$$

which is clearly $1/m_P$ down. However,

$$k_\mu D^{\mu\nu}(k, v) = \frac{k^\nu - v^\nu v \cdot k}{(v \cdot k)^2}. \quad (13)$$

We can disregard the k^μ piece by the usual argument, to find

$$k_\mu D^{\mu\nu}(k, v) \sim -\frac{v^\nu}{v \cdot k} \approx -\frac{2m_P v^\nu}{\vec{k}^2}. \quad (14)$$

While (12) is indeed $1/m_P$ down it may not be neglected as the photon propagator is not just of order 1 but there are components of it that are of order m_P .

The upshot of this discussion is that one cannot simply apply the heavy quark limit blindly. We have seen that while one is, at the level that we are working, allowed to use the covariant gauges one cannot employ the axial gauge without mixing different orders in the heavy quark (proton) mass.

1.3 Physics of Flavour Changing Transitions

We will be interested, in these notes, in calculating the flavour changing electro-weak transitions of one heavy hadron into another. The transitions of prime interest will be where the heavy quark in the first interacts with the electro-weak particle to flavour change into the heavy quark of the second. The light quarks will essentially be spectators. The reason for this is that in the heavy mass limit the heavy quark, as we have seen, is essentially on-shell and acts as a colour source for the light degrees of freedom. In particular the spin of the heavy quark decouples from the dynamics. Thus the dynamics of the weak transition of a heavy hadron are essentially determined by the point like interaction of the weak current with the heavy quark. The picture which emerges of the transition is as follows.

In the infinite mass limit in the rest frame of the heavy hadron, the heavy quark Q is at rest and is surrounded by the light cloud with no spin interaction between them. In the transition the heavy quark Q emits a W meson and becomes another heavy quark Q' moving with some velocity (the velocity of the final heavy hadron). The light cloud has thus to adjust its velocity to keep up with the heavy quark Q' in forming the new hadron. It is this adjustment, or overlap, of the light degrees of freedom which gives rise to the form factors. We immediately see from this simple, atomic physics, picture the physical reason why the unique form factor in heavy meson decays, for example, is normalised to one at zero recoil (or q_{max}^2). At this kinematical point, the daughter heavy quark is produced at rest in the initial rest frame. It is clear that nothing has changed as far as the light degrees of freedom are concerned because there is no flavour dependence in the static colour force due to the heavy quark and the light degrees of freedom do not feel the effect of the change in the heavy quark mass (both are infinite) unless it

moves. Thus there is a complete overlap of the light wavefunctions before and after the transition at this kinematical point and hence the form factor is one. Put bluntly there is no ‘dynamics’ in the transition at this point.

2 Effective Theory to Lowest Order

It is our aim in this section to derive an effective field theory from QCD which captures the essence of the heavy mass limit. In this limit the mass of the heavy hadron, the bound state¹, is approximately the mass of its heavy constituents, but it is important to remember that these do not make up the total momentum of the heavy hadron; although taken to be nearly massless, the light degrees of freedom are allowed to carry off momentum. One is tempted to say that, if there is just one heavy quark inside the hadron, the 4-velocity of the bound state is the 4-velocity of its heavy quark. This is not quite true, but something can be said about the heavy quarks 4-momentum projected in the direction of flight of the heavy hadron, as will be made concrete below.

2.1 Basics of the Effective Theory

Georgi [8] describes the situation rather vividly. If the bound state, made up of a light quark system and one heavy quark, is moving with velocity v^μ ($v^2 = 1, v^0 > 0$), then the 4-momentum of the bound state is

$$P_Q^\mu = M_Q v^\mu = m_Q v^\mu + k^\mu, \quad (15)$$

where M_Q is the mass of the bound state, essentially the same as the mass of the heavy quark m_Q (as indicated by the second equality). Clearly $k^\mu = (M_Q - m_Q)v^\mu$, which in the weak binding limit is $k^\mu = \sum_q m_q v^\mu$ where m_q is the light quark mass. If the bound state undergoes scattering into a new state of the same heavy quark, with momentum

$$P'_Q^\mu = m_Q v'^\mu + k'^\mu, \quad (16)$$

($v'^2 = 1$), the momentum transfer is $m_Q(v - v')^\mu + (k - k')^\mu$. In the limit as $m_Q \rightarrow \infty$, for fixed $k^\mu - k'^\mu$, then necessarily $v'^\mu = v^\mu$ to keep the momentum transfer finite. Conversely, to alter the velocity of the heavy hadron takes an infinite amount of 4-momentum. So one is able to follow the velocity of the heavy hadron. Soft gluons will not alter the heavy bound state velocity. Only very hard gluons or electroweak transitions can do this.

¹Subsequently, unless otherwise stated, whenever we say the heavy particle (or heavy hadron) we mean the bound state containing the heavy quark. M_Q is always the mass of the Q-hadron while m_Q is the Q quark mass.

To sharpen this picture, consider the heavy quark's behaviour during this scattering event. The heavy quark is off shell, but as the physical mass is (in the limit $m_Q \rightarrow \infty$) identifiable with the heavy quark mass, the heavy quark cannot be 'far' off shell. It carries most of the four momentum of the system. Decompose the hadron 4-momentum, before the scattering, as a sum of the heavy quark 4-momentum p_Q and that carried by the light degrees of freedom p_q

$$P_Q = m_Q v + k = p_Q + p_q. \quad (17)$$

The heavy quark four momentum is then $p_Q = m_Q v + (k - p_q)$. Repeating the argument above, now for finite p_q as well, establishes that the heavy quark 4-momentum after the scattering event is $p'_Q = m_Q v + (k' - p'_q)$ (p'_q finite).

From this we learn that the velocity of the heavy quark is *not* fixed, but rather the projection of the heavy quark's four momentum in the direction of the four momentum of the hadron is the heavy quark mass, as for finite $(k - p_q)$, $v.(k - p_q)$ is very small compared to the mass. Thus

$$v.p_Q \simeq m_Q. \quad (18)$$

The idea then, is to construct an effective field theory that is able to keep track of heavy hadrons with *given* velocity and which also encodes the fact that the heavy quark momentum satisfies (18) "on shell".

Georgi [8], following Eichten and Hill [5], did precisely this in the limit that the heavy quark masses are infinite. The action that he finds in this case is

$$S_Q^0(v) = \int \bar{Q}(i\gamma^\mu v.D - m_Q)Q, \quad (19)$$

for a given heavy quark with a given hadron velocity v^μ , and with the covariant form of the projection (18) as the kernel², where D is the covariant derivative $D_\mu = \partial_\mu - igA_\mu$. Here $A = A^a T^a$ and T^a are the usual Hermitian SU(3) Lie algebra generators. The heavy quark effective theory at this order will be called HQET₀.

We have seen, in the introduction, that the heavy quark propagator can be expanded as

$$\frac{i}{\not{p}_Q - m_Q} = i \frac{(1 + \gamma^5)/2}{v.k} + O(k/m_Q). \quad (20)$$

This is precisely what one gets for the zeroth order term on expanding, in powers of $O(k/m_Q)$, the tree-level heavy quark propagator

$$\frac{i}{\not{\gamma}v.p_Q - m_Q} \quad (21)$$

² It is not quite this as we have not rotated out the mass, nor have we summed over all possible velocities, but the actions are essentially equivalent.

arising from the effective action (19). In fact one can also write

$$\begin{aligned} \frac{i}{\not{v} \cdot \not{p}_Q - m_Q} &= i \frac{\frac{1}{2}(1 + \not{v})}{\not{v} \cdot \not{p}_Q - m_Q} - i \frac{\frac{1}{2}(1 - \not{v})}{\not{v} \cdot \not{p}_Q + m_Q} \\ &= i \frac{\frac{1}{2}(1 + \not{v})}{\not{v} \cdot \not{k}} - i \frac{\frac{1}{2}(1 - \not{v})}{\not{v} \cdot \not{k} + 2m_Q}. \end{aligned} \quad (22)$$

and then one notes that the heavy quark mass pole is in the first term and as $m_Q \rightarrow \infty$ the second term goes further and further away from the pole. All the higher order, in $(1/m_Q)$, terms arise from the second factor in the propagator (22). Subsequently in Green's functions when considering terms of zeroth order we shall often take the heavy quark propagator to consist only of the first term in the above equation (22).

Similarly, the 3-point function (a heavy quark and a gluon) is given by

$$G_\mu^{(2,1)}(p_Q, q) = \frac{i}{\not{p}_Q - m_Q} (ig\gamma^\nu) \frac{i}{\not{p}_Q + \not{q} - m_Q} \Delta_{\nu\mu}(q), \quad (23)$$

where $\Delta_{\nu\mu}(q)$ is the gluon propagator. Now consider soft gluons, i.e. q is also of order Λ_{QCD} . Then we can expand as before to obtain

$$G_\mu^{(2,1)}(p_Q, q) = \left(\frac{1 + \not{v}}{2}\right) \frac{i}{\not{v} \cdot \not{k}} (igv^\nu) \left(\frac{1 + \not{v}}{2}\right) \frac{i}{\not{v} \cdot (\not{k} + \not{q})} \Delta_{\nu\mu}(q) + O(\Lambda_{QCD}/m_Q). \quad (24)$$

Exercise: Check the form of the two and three point functions in the heavy quark limit. Hint:

$$\left(\frac{1 + \not{v}}{2}\right) \gamma^\nu \left(\frac{1 + \not{v}}{2}\right) = \left(\frac{1 + \not{v}}{2}\right) v^\nu \left(\frac{1 + \not{v}}{2}\right). \quad (25)$$

This corresponds precisely to the vertex, $i\not{v}v^\nu$, obtained from the effective Lagrangian (19), when sandwiched between the leading order term in the heavy quark propagator (22). Thus in the limit $m_Q \rightarrow \infty$ the only part of the gluon field which contributes is that along the v -direction. These results can be easily extended to arbitrary tree level diagrams, provided that we have only one heavy quark and all other particles carry small momenta.

The theory defined by (19) has some rather remarkable properties. All of the simple characteristics and enlarged symmetries that the theory enjoys may be traced back to the fact that the operator $(i\not{v}v \cdot D - m_Q)$ that appears in (19) depends only on a one-dimensional derivative and only on one linear combination of the Dirac matrices, namely \not{v} . For all intents and purposes this action defines a one dimensional field theory in the heavy quark sector (whence solvable in that sector). This should be kept in mind.

2.2 Flavour Symmetry

The flavour symmetry at this order, valid in the static limit, may be seen as follows. The combined action for a b and c quark with hadron velocities v_b and v_c respectively is

$$S_b^0(v_b) + S_c^0(v_c) = \int \bar{b}(i\gamma_b v_b \cdot D - m_b)b + \int \bar{c}(i\gamma_c v_c \cdot D - m_c)c . \quad (26)$$

The flavour mixing symmetry is between b and c quarks in hadrons moving with the *same* velocity, so that the relevant form of (26) is

$$S_b^0(v) + S_c^0(v) = \int \bar{b}(i\gamma v \cdot D - m_b)b + \int \bar{c}(i\gamma v \cdot D - m_c)c , \quad (27)$$

where $v = v_b = v_c$. The symmetry transformation is

$$\begin{aligned} \delta c &= \epsilon \exp(i\gamma v \cdot x(m_b - m_c))b, \\ \delta b &= -\epsilon \exp(i\gamma v \cdot x(m_c - m_b))c, \end{aligned} \quad (28)$$

where ϵ is an infinitesimal parameter. If one includes the top quark (or other heavy flavour quarks) there is a corresponding set of transformations leaving the action invariant.

Exercise: Derive the corresponding invariance for F heavy flavours.

In order to be rid of the cumbersome exponentials in (28) one may, if one wishes, define new fields

$$\hat{Q} = \exp(i\gamma v \cdot x m_Q) Q , \quad (29)$$

in terms of which the action (19) has the same form but the mass is set to zero [8]

$$S_Q^0(v) = \int \bar{\hat{Q}}(i\gamma v \cdot D)\hat{Q} . \quad (30)$$

This property that one may define fields in terms of which the mass is transformed away is characteristic of the one-dimensional first order operator. Eq.(28) is the infinitesimal form of a more general global symmetry, here a U(2) symmetry (in general U(F) for F heavy flavours). Thinking of the (b, c) as an U(2) doublet the action

$$\sum_{Q=b,c} \int \bar{\hat{Q}}(i\gamma v \cdot D)\hat{Q} \quad (31)$$

is invariant under

$$(\hat{b}, \hat{c}) \rightarrow (U\hat{b}, U\hat{c}) \quad , \quad \begin{pmatrix} \bar{b} \\ \bar{c} \end{pmatrix} \rightarrow U^\dagger \begin{pmatrix} \bar{b} \\ \bar{c} \end{pmatrix} , \quad (32)$$

where $U \in \text{U}(2)$.

2.3 Particle Properties

A second characteristic of the one dimensional nature of the field theory is the fact that one may create ‘covariant’ velocity projection operators to project out quark and anti-quark states. Let

$$P_{\pm}(v) = \frac{1}{2}(1 \pm \not{v}), \quad (33)$$

with the properties

$$P_+^2 = P_+, \quad P_-^2 = P_-, \quad P_{\pm}P_{\mp} = 0, \quad P_+ + P_- = 1. \quad (34)$$

Now decompose the heavy quark field as

$$Q = Q_+ + Q_-, \quad Q_{\pm} = P_{\pm}Q, \quad (35)$$

so that Q_+ satisfies $P_-Q_+ = 0$, which is the Dirac equation $(\not{v} - m_Q)Q_+ = 0$ for a quark with momentum m_Qv^{μ} and Q_- satisfies the Dirac equation for an anti-quark with the same momentum. As operators these annihilate precisely those states. Note that these projection operators act in much the same way as the parity projection operators do in the rest frame of a ‘normal’ particle, which when boosted become Dirac equations [2]. In any case the action (19) decomposes into quark and anti-quark pieces

$$S_Q^0(v) = \int \bar{Q}_+(iv.D - m_Q)Q_+ - \bar{Q}_-(iv.D + m_Q)Q_-. \quad (36)$$

These projections are also reminiscent of chirality, and do not tell us about the dynamics of the wave equation itself (except of course that it has two orthogonal sets of solutions). The free wave equation for Q_+ , for example, is

$$(iv.\partial - m_Q)Q_+ = 0, \quad (37)$$

with the solutions

$$Q_{+\alpha}(x) = P_{+\alpha}^{\beta}F_{+\beta}(x^{\perp})e^{-im_Qv.x}, \quad (38)$$

where F_+ is an arbitrary Dirac spinor depending only on $x^{\perp\mu} = x^{\mu} - v^{\mu}v.x$ which satisfies $x^{\perp}.\not{v} = 0$, so that $F_{+\beta}(x^{\perp})$ is unaffected by $v.\partial$.

Physically, the situation is then somewhat underdetermined. Certainly the wave function satisfies the Dirac equation (its ‘chirality’ is fulfilled) but what happens orthogonally to the line of flight of the hadron is not known³. The reason for this is not too surprising. A one-dimensional field theory can only

³In previous works we have pointed out, rather euphemistically, that it is the heavy quark ‘label’ that satisfies the Dirac equation with velocity v and not that the heavy quark itself has only that velocity. We hope our remarks here clarify the situation.

determine the dynamics in that direction. There is a similar solution for \tilde{Q}_- with similar conclusions.

An alternative way of saying this is the following. The fact that the wave equation has such arbitrariness in its solution is an indication that the action has a rather large invariance. We may as well exhibit this for (19). Let

$$Q \rightarrow M(x^\perp)Q, \quad \bar{Q} \rightarrow \bar{Q}M^{-1}(x^\perp), \quad (39)$$

with M diagonal in colour space, but a general matrix in spinor space. To guarantee invariance of the action one only needs that $[\psi, M] = 0$, which is partially solved by $M = P_+M_- + P_-M_+$ where M_+ and M_- are functions. These invariances of the action account for the arbitrariness in the wave functions (38). A complete decomposition of the invariances is given in the appendix.

2.4 Covariance and Spin (I)

How covariant is the construction thus far? The actions (19) and (36) depend explicitly on a fixed four-vector, so that arbitrary Lorentz boosts cannot be a symmetry of the action. Having chosen a preferred 4-velocity we wish to consider the subgroup of the Lorentz group which keeps this time-like direction fixed. In the heavy hadron's rest frame $v^\mu = (1, \underline{0})$ the actions are clearly invariant under the $O(3)$ group of three space rotations. Let a general infinitesimal Lorentz transformation be parameterised by $\eta_{\mu\nu} + \lambda_{\mu\nu}$, where $\eta_{\mu\nu}$ is the metric tensor.

Then the little group is defined to be generated by all those $\lambda_{\mu\nu}$ which satisfy

$$\lambda_\mu{}^\nu v_\nu = 0. \quad (40)$$

It is easy to see that, for $\lambda_{\mu\nu}$ satisfying this condition, the heavy quark action is invariant under the transformations

$$\begin{aligned} \delta Q_\pm &= \frac{1}{4} \lambda^{\mu\nu} \gamma_\mu \gamma_\nu Q_\pm, \\ \delta \bar{Q}_\pm &= -\frac{1}{4} \bar{Q}_\pm \lambda^{\mu\nu} \gamma_\mu \gamma_\nu, \end{aligned} \quad (41)$$

with the consistency property that the particle projection operators P_\pm commute with the spin transformation, $[P_\pm, \lambda^{\mu\nu} \gamma_\mu \gamma_\nu] = 0$, so that the transformed fields remain unambiguously eigenstates of P_\pm , i.e.

$$P_\mp \delta Q_\pm = 0. \quad (42)$$

The spin transformations (41) require some explanation. There is no transformation (Lorentz rotation) of the co-ordinates x^μ . This comes about because the operator $(i\psi v.D - m_Q)$, as we noted before, acts only in the direction of v ,

so any Lorentz transformation $x^\mu \rightarrow x^\mu + \lambda^\mu{}_\nu x^\nu$ with $\lambda_\mu{}^\nu v_\nu = 0$ is by itself a symmetry of the action. For the fields this translates into

$$\begin{aligned}\delta Q_\pm(x) &= -\lambda^\mu{}_\nu x^\nu \partial_\mu Q_\pm(x), \\ \delta \bar{Q}_\pm(x) &= -\lambda^\mu{}_\nu x^\nu \partial_\mu \bar{Q}_\pm(x),\end{aligned}\tag{43}$$

$$\delta A_\rho = -\lambda^\mu{}_\nu x^\nu \partial_\mu A_\rho\tag{44}$$

which is a symmetry of the action. The two sets of transformations (41) and (43) together form the conventional Lorentz transformations for the spinors.

Exercise: Check the invariance of the heavy quark action (19) under the above transformation rules.

Note also that the transformations (41), the solutions given after (39), and

$$\delta Q = \gamma_5 \not{e}^\perp(v) M_0 Q, \quad \delta \bar{Q} = \bar{Q} \not{e}^\perp(v) \gamma_5 M_0, \tag{45}$$

with M_0 a function of x^\perp and $\not{e}^\perp(v) = \not{e} - \not{v} \cdot \not{e}$ ⁴ exhaust the symmetry of the action (19) in four dimensions. Indeed, the transformation (41) may also be generalized as

$$\begin{aligned}\delta Q &= \not{\eta}^\perp(v) \gamma_5 \not{e} Q M_1, \\ \delta \bar{Q} &= -\bar{Q} \not{\eta}^\perp(v) \gamma_5 \not{e} M_1,\end{aligned}\tag{46}$$

where η is defined by $\lambda^{\mu\nu} \gamma_\mu \gamma_\nu = \not{\eta}^\perp(v) \gamma_5 \not{e}$ and M_1 is a function.

2.5 Field Choices and Interactions

As only one component of the gauge field appears in the action $S_Q^0(v)$, namely $v \cdot A$, if we pick the gauge $v \cdot A = 0$ then the heavy quarks associated with that hadron velocity decouple from the glue. Such non-covariant gauges may be quite difficult to work with, and in any case are of limited use for they provide no simplification for heavy particles with other velocities. In fact, as we saw in the introduction, this gauge can, in certain situations be dangerous⁵. We would like to see this decoupling, therefore, in a more general fashion. Just as it was possible to define fields for which the mass does not appear in the action (30), we can

⁴For a general tensor of rank n , A_{μ_1, \dots, μ_n} , we set $A_{\mu_1, \dots, \mu_n}^\perp(v) = \eta_{\mu_1}^{\perp\nu_1}(v) \dots \eta_{\mu_n}^{\perp\nu_n}(v) A_{\nu_1, \dots, \nu_n}$ with $\eta_{\mu\nu}^\perp(v) = \eta_{\mu\nu}^\perp - v_\mu v_\nu$.

⁵Though for properties of transitions at equal velocity, that is at maximum momentum transfer, such gauges have been very useful [23].

similarly define fields \tilde{Q} and $\bar{\tilde{Q}}$ for which the gauge interaction is absent. These are given by

$$\begin{aligned} Q(x) &= W \begin{bmatrix} x \\ v \end{bmatrix} \tilde{Q}(x) \\ \bar{Q}(x) &= \bar{\tilde{Q}}(x) W \begin{bmatrix} x \\ v \end{bmatrix}^{-1}, \end{aligned} \quad (47)$$

where the Wilson line

$$W \begin{bmatrix} x \\ v \end{bmatrix} = P \exp \left[ig \int_{-\infty}^{v \cdot x} ds A \cdot v \right]$$

is a path-ordered exponential, wherein the path is a straight line from ∞ to x along the v -direction.

In terms of these fields (19) becomes

$$S_Q^0(v) = \int \bar{\tilde{Q}}(i \not{v} \cdot \partial - m_Q) \tilde{Q} . \quad (48)$$

So S_Q^0 represents free heavy quarks in any gauge. Let us explain how this comes about. To say we can pick a gauge $v \cdot A = 0$ is to say that given an arbitrary gauge field A there exists a gauge transformation, which takes us to that gauge. Specifically, if

$$A' = \xi^{-1} A \xi + \xi^{-1} \frac{i}{g} \partial \xi, \quad (49)$$

and $v \cdot A' = 0$ then

$$0 = \xi^{-1} (v \cdot A + \frac{i}{g} v \cdot \partial) \xi, \quad (50)$$

is an equation for ξ , solved by

$$\xi(v \cdot x) = W \begin{bmatrix} x \\ v \end{bmatrix} \quad (51)$$

with the boundary condition that, at $v \cdot x = -\infty$, ξ is the identity (actually we could just as well take (50) to be the defining equation of the path-ordered exponential). Substituting (47) into (19) we find

$$\begin{aligned} S_Q^0(v) &= \int \bar{\tilde{Q}}(i \not{v} \cdot \partial - m_Q) \tilde{Q} \\ &= \int \bar{\tilde{Q}} \xi^{-1} (i \not{v} \cdot \partial - m_Q) \xi \tilde{Q} \\ &= \int \bar{\tilde{Q}}(i \not{v} \cdot \partial - m_Q) \tilde{Q} + i \int \bar{\tilde{Q}} \not{v} \xi^{-1} ((v \cdot \partial \xi) - ig v \cdot A \xi) \tilde{Q} \\ &= \int \bar{\tilde{Q}}(i \not{v} \cdot \partial - m_Q) \tilde{Q}. \end{aligned} \quad (52)$$

We see that, in changing variables according to (47), the gauge field A is altered to the gauge field A' which satisfies the gauge condition $v \cdot A' = 0$, giving the

triviality of the action. So there is no gauge condition on A . It is just that the combination which makes up A' satisfies (50). The fields \tilde{Q} are gauge inert, meaning that they do not vary under gauge transformations, as can be seen from their definition. This ensures that the action (52) is gauge invariant as it should be.

Let us note that this is rather an extreme situation. The effective theory is totally disinterested in the coupling of gluons to the heavy quarks. This shows us that the action (19) really models the situation described above, namely, that while keeping track of the heavy particle velocity, soft gluons are not important for questions concerning transitions.

In terms of the effective fields (47) the action (19) of HQET₀ takes the free form described above (52). The symmetry (43) is thus directly generalised to the free case by

$$\delta\tilde{Q}(x) = -\lambda^\mu{}_\nu x^\nu \partial_\mu \tilde{Q}(x) , \quad \delta\bar{\tilde{Q}}(x) = -\lambda^\mu{}_\nu x^\nu \partial_\mu \bar{\tilde{Q}}(x) . \quad (53)$$

For completeness, let us note that the Feynman propagator for the non-interacting fields \tilde{Q} is

$$\begin{aligned} \tilde{S}_F(x-y; v) = & -i\frac{(1+\gamma)}{2}\theta(v.x - v.y)\delta_\perp^3(x-y)e^{-im_Q(v.x-v.y)} \\ & -i\frac{(1-\gamma)}{2}\theta(v.y - v.x)\delta_\perp^3(y-x)e^{-im_Q(v.y-v.x)} \end{aligned} \quad (54)$$

where the argument of the delta function $\delta_\perp^3(x)$ is the component of x not in the v -direction. To pass to the Feynman propagator for the interacting field Q is particularly simple in view of (47),

$$S_F(x, y; v; A) = S_F(x-y; v)W\begin{bmatrix} x \\ v \end{bmatrix}W\begin{bmatrix} y \\ v \end{bmatrix}^{-1} . \quad (55)$$

Notice that these only propagate quarks forward in “time” (we mean $v.x$ which is the usual time component in the rest frame) while only anti-quarks propagate backwards in “time”. One immediate consequence of this is that there are no heavy quark loops in the HQET₀, as could be inferred directly from the fact that they are certainly not there in the $v.A = 0$ gauge.

Exercise: Show that in QED one can calculate any Green’s function involving \tilde{Q} and A_μ non-perturbatively by going over to the free spinor variable Q . Hint: Calculate a Wilson loop in pure Maxwell theory. Why does this procedure not work in QCD ?

2.6 Derivation of Lowest Order Action

Questions that naturally arise are, how does one get to the effective action (19) directly from QCD and what are the corrections that are needed for finite m_Q ?

As recognised in [25], this action is familiar in a slightly different but related context. Namely it is a generalisation of the so-called ‘Bloch-Nordsieck’ model [31, 32] that one arrives at in studying the infrared behaviour of QED as the electron goes on shell. The similarities of the two contexts in which this action arises will not have escaped the reader’s attention. Indeed we know how to derive the non-relativistic action, and we may use this knowledge to get at the heavy quark effective theory.

Recall that to obtain the relevant non-relativistic information about the Dirac equation one makes use of Foldy-Wouthuysen transformations. These are canonical unitary transformations which eliminate the interaction terms between the positive and negative frequency components of the fermion spinors in the Hamiltonian. A peek at (19,36), and the subsequent discussion, shows us that this is precisely the situation that we would like to get to.

In field theory the analogous set of transformations that give a systematic derivation of the non-relativistic limit of the relativistic Hamiltonian in non-Abelian theories was presented by Feinberg [6] (see also [33, 7]). The non-relativistic limit picks out a preferred frame (usually the rest frame, but this need not be so). Likewise, in the heavy quark limit, one would like to fix or pick out the “velocity” of the b - (or other heavy) quark that matches that of the b - (or heavy) hadron. Furthermore we would like to arrive at a systematic expansion (in the inverse mass) giving corrections to this picture. This is achieved by making use of the Foldy-Wouthuysen transformations in the context of Lagrangian field theory.

Here we derive (19) by ignoring $1/m_Q$ corrections, which will be taken up again in later sections. Consider the action

$$S_Q = \int \bar{\psi}_Q (i\mathcal{D} - m_Q) \psi_Q , \quad (56)$$

where the subscript Q , as before, indicates a heavy quark. We make no distinction here between renormalized and unrenormalized quantities, as it is not relevant for our present purposes and would clutter up the formulae. Following [23] we perform the change of variables

$$\begin{aligned} \psi_Q &= e^{i(\mathcal{D} - \not{v} \cdot \mathcal{D})/(2m_Q)} Q \\ &= Q + i(\mathcal{D} - \not{v} \cdot \mathcal{D})/(2m_Q) Q + O(1/m_Q^2) + \dots , \\ \bar{\psi}_Q &= \bar{Q} e^{-i(\mathcal{D} - \not{v} \cdot \mathcal{D})/(2m_Q)} \\ &= \bar{Q} - \bar{Q} i(\mathcal{D} - \not{v} \cdot \mathcal{D})/(2m_Q) + O(1/m_Q^2) + \dots , \end{aligned} \quad (57)$$

where \overleftarrow{D} is defined by

$$\int f \overleftarrow{D} g = - \int f \vec{D} g . \quad (58)$$

The Jacobian of this transformation within dimensional regularization has been shown to be unity in [23]. We stay within the realm of dimensional regularization in the following.

The action may now be expressed as:

$$\begin{aligned}
S_Q &= \int \bar{Q} [1 + i(\not{P} - \not{v} \cdot \not{D})/(2m_Q)] (i\not{P} - m_Q) [1 + i(\not{P} - \not{v} \cdot \not{D})/(2m_Q)] Q \\
&= \int \bar{Q} (i\not{v} \cdot \not{D} - m_Q) Q + O(1/m_Q) + \dots \\
&= S_Q^0(v) + O(1/m_Q) + \dots,
\end{aligned} \tag{59}$$

which is the relationship we have been looking for. A feature that has just been used and that will persist throughout is that the mass in (56) lowers the order of the terms in the transformation (57), allowing for the replacement of the covariant derivative in (56) with the one-dimensional covariant derivative.

The reason for the particular exponential form of the Foldy-Wouthuysen transformation will be explained in detail when we tackle the derivation of the effective theory at $O(1/m_Q)$. For now, we note that to really be able to denote the extra terms in (59) as $O(1/m_Q)$ we must be sure that there are no components of the derivatives that appear that lie in the v direction. This is exactly what the form of the Foldy-Wouthuysen transformations (57) guarantees.

In this derivation we have assumed that both, the components of the momentum of the heavy quark, and the components of the gauge field that are orthogonal to the line of flight are small relative to the (infinite) heavy quark mass. For this to be the case, the interactions must be such that there are no hard gluons involved.

2.7 Covariance and Spin (II)

While, as we have discussed above, the effective action (19) is invariant under little group transformations it is clearly *not* invariant under general Lorentz transformations,

$$\begin{aligned}
\delta_\lambda Q(x) &= L_\lambda Q(x) \\
&= (-\lambda^\mu{}_\nu x^\nu \partial_\mu + \frac{1}{4} \lambda^{\mu\nu} \gamma_\mu \gamma_\nu) Q(x), \\
\delta_\lambda \bar{Q} &= \bar{L}_\lambda \bar{Q} \\
&= -\lambda^\mu{}_\nu x^\nu \partial_\mu \bar{Q} - \frac{1}{4} \bar{Q} \lambda^{\mu\nu} \gamma_\mu \gamma_\nu \\
\delta_\lambda A_\rho &= -\lambda^\mu{}_\nu x^\nu \partial_\mu A_\rho + \lambda_\rho{}^\nu A_\nu.
\end{aligned} \tag{60}$$

This circumstance prompted Georgi [8] to integrate over all possible velocities, with (19) as the integrand, so as to restore manifest covariance. Somewhat later this integral over velocities was demoted to a sum although it was not clear what

the domain of the sum should be. The derivation, presented in the last section, of the effective theory directly from the *QCD* action, fixes once and for all the four-velocity to that of the physical hadron.

The non-covariance of the action (19) under the Lorentz transformation (60) is actually a desired feature. This may be seen from two points of view. Firstly, think of the Lorentz transformation in the active sense, that is, as a boost of the hadron from its old four velocity v to a new four velocity $v' = v + \lambda \cdot v$. The effective action designed to describe the new situation is $S_Q^0(v')$, which is precisely what one obtains under the transformation (60), namely $S_Q^0(v') = S_Q^0(v) - \delta_\lambda S_Q^0(v)$. There is no transformation on v in the variation. This was, after all, to be expected.

What we really want are the usual covariance identities that arise from Lorentz invariance; for example (with $v'^\mu = \Lambda^\mu{}_\nu v^\nu = v^\mu + \lambda^\mu{}_\nu v^\nu$)

$$\langle H_c(v'_c) | [\bar{c} J_\mu \tilde{b}] (0) | H_b(v'_b) \rangle = \Lambda_\mu{}^\nu \langle H_c(v_c) | [\bar{c} J_\nu \tilde{b}] (0) | H_b(v_b) \rangle. \quad (61)$$

The velocity dependence on the right hand side rests in the action $S_Q^0(v_Q)$. Perform a Lorentz transformation on all the fields with parameter $\lambda^\mu{}_\nu$ on the right hand side. This is just a change of dummy variables in the path integral approach. The states $|H_Q(v_Q)\rangle$ map to $|H_Q(v'_Q)\rangle$, while the action, as we saw, becomes $S_Q^0(v'_Q)$. Thus (61) is established.

Now to the second point of view. The effective action taken to all orders is equivalent to the *QCD* action, which is covariant. Hence covariance must be achieved in the full effective theory. But incorporating higher order (in $1/m_Q$) terms in the action would hardly help the situation. The missing piece is that (60) is *not* the correct transformation rule for the effective fields! If $\mathcal{D}^\perp(v) = \mathcal{D} - \not{v} \cdot D$ transformed correctly as a bi-spinor then the effective fields would have the transformation rule (60). However, as v is a fixed vector, we find, using the defining equation (57), to first order in $1/m_Q$, that

$$\begin{aligned} \delta_\lambda Q(x) &= L_\lambda Q(x) + \frac{i}{2m_Q} [v \cdot \lambda \cdot \gamma v \cdot D + \not{v} \cdot \lambda \cdot D] Q(x), \\ \delta_\lambda \bar{Q} &= \bar{L}_\lambda \bar{Q} - \frac{i}{2m_Q} \bar{Q} [v \cdot \lambda \cdot \gamma v \cdot \bar{D} + \not{v} \cdot \lambda \cdot \bar{D}]. \end{aligned} \quad (62)$$

Here $a \cdot \lambda \cdot b = a_\mu \lambda^{\mu\nu} b_\nu$. It is straightforward to establish that (62), alongwith the usual transformation of the gauge field, leaves the lowest order action (19) invariant upto order $1/m_Q$. Taking into account higher order terms in both the action and the transformation allows one to establish covariance at any given order in the inverse mass.

Exercise: Derive (62). For fun determine the $1/m_Q^2$ corrections to the transformations.

3 Current Induced Transitions

For definiteness consider the transition of a bound state containing one b -quark $|\Phi_b\rangle$ to another bound state containing one c -quark $|\Phi_c\rangle$, induced by a flavour changing current $J_\mu \equiv V_\mu - A_\mu = \bar{\psi}_c \gamma_\mu (1 - \gamma_5) \psi_b \equiv \bar{\psi}_c \Gamma_\mu \psi_b$. Then we can write the transition-matrix element as

$$\langle \Phi_c | J_\mu | \Phi_b \rangle . \quad (63)$$

The situation may be described as follows. As the b hadron comes in from the far past, we are able to track its four velocity v_b , and after the transition we may follow the c -hadron's four-velocity v_c into the future. In this way the relevant action to consider is (26)

$$S_b^0(v_b) + S_c^0(v_c) = \int \bar{b}(i\gamma_b v_b \cdot D - m_b)b + \int \bar{c}(i\gamma_c v_c \cdot D - m_c)c , \quad (64)$$

which is repeated here for convenience.

Let us embroider on the picture of the heavy quark theory that has been developed thus far. A bound state made up of one heavy quark and some light quarks moving with velocity v , such that the 4-momentum of the system is

$$P^\mu = M_Q v^\mu , \quad (65)$$

undergoes a current induced transition to a heavy bound state of a different flavour, with a new four velocity v' and four momentum

$$P'^\mu = M_{Q'} v'^\mu . \quad (66)$$

Then the momentum transfer, $q^\mu = M_Q v^\mu - M_{Q'} v'^\mu$; in the limit as the heavy quark masses go to infinity it diverges. The square of the momentum transfer is $q^2 = q_{max}^2 - 2M_Q M_{Q'}(v \cdot v' - 1)$ and is bounded by $q_{max}^2 = (M_Q - M_{Q'})^2$, which is achieved at equal velocity $v = v'$. Of course, even though q has components that are large, q^2 may well be small e.g. $q^2 = 0$ at $v = (1, \underline{0})$ and $v' = (\frac{M_Q^2 + M_{Q'}^2}{2M_Q M_{Q'}}, \underline{v'})$ with $\underline{v'} = (\frac{M_Q^2 - M_{Q'}^2}{2M_Q M_{Q'}}, 0, 0)$ say. The velocities are well defined when the ratio $M_Q/M_{Q'}$ is finite, which we take to be the case.

3.1 LSZ Reduction and the Effective Theory

To get a handle on the transition (63) we first make use of the ideas of the reduction theorems. These tell us that as long as we can find an ‘interpolating field’ $\phi(x)$ corresponding to a state $|\Phi(P)\rangle$ with the property (for the S -wave pseudoscalar state)

$$\langle 0 | \phi(x) | \Phi(P) \rangle = e^{-iP \cdot x} , \quad (67)$$

then the transition may be expressed as⁶

$$L(P_b, P_c) \int d^4x d^4y e^{iP_c \cdot x} e^{-iP_b \cdot y} \langle 0 | \phi_c(x) J_\mu(0) \phi_b^\dagger(y) | 0 \rangle, \quad (68)$$

where

$$L(P_b, P_c) = \lim_{P_b^2 \rightarrow M_b^2} \lim_{P_c^2 \rightarrow M_c^2} (P_b^2 - M_b^2)(P_c^2 - M_c^2).$$

We shall concern ourselves with the exact form of the interpolating fields ϕ_Q presently. For now we note that whatever their precise form, they will have the structure

$$\phi_Q(x) = N_{\Phi_Q} \bar{\chi}_\beta^\alpha(\partial) T \bar{\psi}_q^{i\beta}(x) \psi_{Q\,i\alpha}(x), \quad (69)$$

$$\phi_Q^\dagger(x) = N_{\Phi_Q} \chi_\alpha^\beta(\partial) T \bar{\psi}_Q^{i\alpha}(x) \psi_{q\,i\beta}(x), \quad (70)$$

with $\bar{\chi} = \gamma_0 \chi^\dagger \gamma_0$ for heavy mesons, while for heavy baryons one has

$$\phi_Q(x) = N_{\Phi_Q} \epsilon^{ijk} \bar{\chi}^{\alpha\beta\gamma}(\partial) T \psi_{Q\,i\alpha}(x) \psi_{q\,j\beta}(x) \psi_{q'\,k\gamma}(x), \quad (71)$$

$$\phi_Q^\dagger(x) = N_{\Phi_Q} \epsilon_{ijk} \chi_{\alpha\beta\gamma}(\partial) T \bar{\psi}_q^{i\gamma}(x) \bar{\psi}_q^{j\beta}(x) \bar{\psi}_Q^{k\alpha}(x), \quad (72)$$

with $\bar{\chi} = \chi^\dagger \gamma_0 \gamma_0 \gamma_0$, where $\chi^{\alpha\beta\dots}$ are projection operators (Dirac matrices possibly including differentiations, total and partial, like $\overset{\leftrightarrow}{\partial}$) which pick out the particle states of interest⁷. The normalization constant N_{Φ_Q} , for brevity denoted by N_Q , is fixed by (67). ψ_Q and ψ_q are the heavy and light quark fields, respectively.

To simplify the formulae we concentrate on mesonic transitions, i.e. from a b -meson to a c -meson. The generalisation to baryons is immediate. One may rewrite (68) in this case as

$$N_b N_c L \int d^4x d^4y e^{iP_c \cdot x} e^{-iP_b \cdot y} \langle 0 | \bar{\chi}_\sigma^\alpha(\partial_x) \bar{\psi}_q^\sigma(x) \psi_{c\alpha}(x) J_\mu(0) \chi_\beta^\rho(\partial_y) \bar{\psi}_b^\beta(y) \psi_{q\rho}(y) | 0 \rangle. \quad (73)$$

Now any total differentiations with respect to x and y , present in $\chi(\partial_y)$ and $\bar{\chi}(\partial_x)$, can be shifted on to the exponentials by partial integration and, calling the subsequent projection operators $\chi(v_b)$ and $\bar{\chi}(v_c)$, one can write the matrix element as

$$N_b N_c L(P_b, P_c) \int d^4x d^4y e^{iP_c \cdot x} e^{-iP_b \cdot y} \langle 0 | \bar{\psi}_b(y) \cdot \chi(v_b) \cdot \psi_q(y) \bar{\psi}_q(x) \cdot \bar{\chi}(v_c) \cdot \psi_c(x) J_\mu(0) | 0 \rangle, \quad (74)$$

⁶All the matrix elements are described in terms of path integrals.

⁷Here, for once, we explicitly show the trace over the colour indices i, j, k . They will be implicit usually.

From now on (.) also indicates sum over Dirac indices, which should be clear from the context. Since we will nearly always consider interpolating fields inside reduction formulae with the relevant exponentials, we may as well always replace $\chi(\partial)$ with $\chi(v)$ in the interpolating field from the beginning. From now on we shall do so unless otherwise stated. Thus far we have followed the full theory in conventional form. Now we apply the effective field theory. In order to do so, we need to assume, that in the complete theory, the integral in (74) will factor according to

$$\frac{1}{(P_b^2 - M_b^2)} \frac{1}{(P_c^2 - M_c^2)} \left(\eta^0 + \frac{1}{2m_c} \eta_c^1 + \frac{1}{2m_b} \eta_b^1 + O(\frac{1}{m_Q^2}) + \dots \right), \quad (75)$$

where η^0 is the term that arises in the lowest order effective theory and η_c^1 and η_b^1 denote first order contributions. Let us naively apply the effective field theory. In (74) replace everywhere the heavy quark fields ψ_Q with the corresponding fields in the effective theory $Q(x) = W \begin{bmatrix} x \\ v_Q \end{bmatrix} \tilde{Q}(x)$ to give

$$N_b N_c L(P_b, P_c) \int d^4x d^4y e^{iP_c \cdot x} e^{-iP_b \cdot y} \langle 0 | \bar{b}(y) \cdot \chi(v_b) \cdot M(y, x) \cdot \bar{\chi}(v_c) \cdot \tilde{c}(x) \bar{\tilde{c}}(0) \cdot \Gamma_\mu \cdot \tilde{b}(0) | 0 \rangle, \quad (76)$$

with

$$M(y, x)_\beta^\alpha = \text{Tr} \psi_{q\beta}(y) \bar{\psi}_q^\alpha(x) W \begin{bmatrix} x & 0 & 0 & y \\ v_c & v_c & v_b & v_b \end{bmatrix}, \quad (77)$$

where the trace is over the colour indices. Here we have defined the matrix product

$$W \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \\ v_1 & v_2 & v_3 & v_4 \end{bmatrix} = W \begin{bmatrix} x_1 \\ v_1 \end{bmatrix} W \begin{bmatrix} x_2 \\ v_2 \end{bmatrix}^{-1} W \begin{bmatrix} x_3 \\ v_3 \end{bmatrix} W \begin{bmatrix} x_4 \\ v_4 \end{bmatrix}^{-1}. \quad (78)$$

We have used the fact that the heavy quark propagators are diagonal in colour. Note that $M(y, x)$ depends explicitly on v_b and v_c . As the effective fields \tilde{Q} decouple from the interactions, (76) factorises into

$$- N_b N_c L(P_b, P_c) \int d^4x d^4y e^{iP_c \cdot x} e^{-iP_b \cdot y} \text{Tr} \chi(v_b) \langle 0 | M(y, x) | 0 \rangle. \\ \bar{\chi}(v_c) \langle 0 | \tilde{c}(x) \bar{\tilde{c}}(0) | 0 \rangle \Gamma_\mu \langle 0 | \tilde{b}(0) \bar{\tilde{b}}(y) | 0 \rangle. \quad (79)$$

Set the momentum through $\langle 0 | M(y, x) | 0 \rangle$ to be q and k . Then the transition is easily expressed as

$$N_b N_c L(P_b, P_c) \int d^4k d^4q \text{Tr} M(q, k) \\ \bar{\chi}(v_c) \frac{1}{\not{v}_c v_c \cdot (P_c - k) - m_c} \Gamma_\mu \frac{1}{\not{v}_b v_b \cdot (P_b + q) - m_b} \chi(v_b), \quad (80)$$

where $M(q, k)$ is defined through the Fourier transform

$$\langle 0 | M(y, x) | 0 \rangle = \int d^4q d^4k e^{-iq \cdot y} e^{-ik \cdot x} M(q, k), \quad (81)$$

and we have introduced the heavy quark propagator

$$\langle 0 | \tilde{Q}(x) \bar{\tilde{Q}}(y) | 0 \rangle = \int d^4 p \frac{e^{-ip.(x-y)}}{\not{v} \cdot p - m_Q}. \quad (82)$$

3.2 Taking the on-shell limit

To ensure that the transition (80) does not vanish there must be poles generated by the integrals that match the zeros that arise from the factors outside the integral. This is an assumption about bound states that needs to be made in the full theory before the effective theory is invoked. However, this need not be taken for granted within the context of the effective field theory, as we have mentioned around (75); rather, due to the simplifications encountered we may and will address this issue in some detail.

We are not free to take the limit in any fashion we choose. For example, let us set $P_Q = M_Q v_Q + \epsilon_Q^\perp(v_Q)$, for some infinitesimal ϵ_Q , and understand the limit to be

$$L(P_Q) = \lim_{P_Q^2 \rightarrow M_Q^2} (P_Q^2 - M_Q^2) = \lim_{\epsilon_Q \rightarrow 0} (\epsilon_Q^\perp)^2.$$

(Recall that $\epsilon^{\perp\mu}(v) = \epsilon^\mu - v^\mu v \cdot \epsilon$ so that $v \cdot \epsilon^\perp = 0$.) The problem with taking the limit in this fashion is that ϵ_Q^\perp cannot appear anywhere in (80) except in $L(P_Q)$! The mass pole is not regulated in this way. An immediate consequence is that we have no choice but to take the limit along the v_Q direction. We then take the limit in the direction $P_Q = M_Q(1 + \epsilon_Q)v_Q$, so that

$$\lim_{P_Q^2 \rightarrow M_Q^2} (P_Q^2 - M_Q^2) = \lim_{\epsilon_Q \rightarrow 0} 2\epsilon_Q M_Q^2. \quad (83)$$

We can leave the cancellation of the zeros implicit in (80), but nevertheless simplify its form. The explicit form of $\chi(v)$, that will be given in the next section, satisfies $(\not{v} - 1)\chi(v) = 0$. The propagators as contracted against the wave functions then take the form

$$(\not{v}_Q v_Q \cdot (P_Q - k) - m_Q)^{-1} \chi \rightarrow \frac{1}{v_Q \cdot (P_Q - k) - m_Q} \chi, \quad (84)$$

so that the transition is quite generally

$$N_b N_c L(P_b, P_c) \int d^4 k d^4 q \frac{1}{v_c \cdot (P_c - k) - m_c} \frac{1}{v_b \cdot (P_b + q) - m_b} \\ Tr M(q, k) \bar{\chi}(v_c) \Gamma_\mu \chi(v_b). \quad (85)$$

There is, however, a way to take the limit so that the expected cancellation of the pole and zero may be explicitly achieved. This requires that the heavy quark

propagator $(v_Q \cdot (P_Q - k) - m_Q)^{-1}$ exactly cancels the zero $(P_Q^2 - M_Q^2)$. This requires that the Green's function $\langle 0 | \tilde{M}(q, k) | 0 \rangle$ is peaked, within the integral, at $v_c \cdot k = \bar{\Lambda}_c$ and $-v_b \cdot q = \bar{\Lambda}_b$, in the heavy quark limit, where $\bar{\Lambda}_Q = M_Q - m_Q$. Then, for example, the c-quark propagator becomes $(v_c \cdot P_c - m_c - \bar{\Lambda}_c)^{-1}$, exactly cancelling the zero $(P_c^2 - M_c^2)$. These conditions are precisely the ‘on-shell’ assumptions that we made in determining the heavy quark theory in the first place. We assumed in (18) that the projection of the quark momentum in the direction of the hadron 4-velocity is the quark mass and in this limit the hadron mass. If this cancellation is accepted then (80) becomes

$$4N_b N_c M_b M_c \int d^4 k d^4 q \text{Tr} M(q, k) \bar{\chi}(v_c) \Gamma_\mu \chi(v_b). \quad (86)$$

It is not mandatory that the cancellation of the poles proceeds in this way. For our purposes it is enough to notice that the denominators of the heavy quark propagators depend only on $\bar{\Lambda}_Q$ and on $\epsilon_Q M_Q$. However, we know (see Introduction) that to lowest order $\bar{\Lambda}_b = \bar{\Lambda}_c = \bar{\Lambda}$. So that after the limits are taken the only dependence will be on $\bar{\Lambda}$, which is universal.

A problem that still remains is, how do the light degrees of freedom know that there is such a momentum restriction (i.e. $v_Q \cdot k \approx \bar{\Lambda}$)? As we have noted the information that is lost in going to the effective fields has to do with how the light degrees of freedom respond to the heavy quark. The discussion thus far is then still not complete. We need to ask ourselves about the applicability of the effective theory in such a naive fashion. Certainly at the point of interaction with the current the only important scale is that set by the transition, and the heavy quark operators may be replaced with their effective counterparts. But that still leaves the heavy quark operators that go into making up the interpolating field.

We parameterise this ignorance by the substitution

$$\chi_{\alpha\alpha_1\dots\alpha_n} \rightarrow \chi_{\alpha\beta_1\dots\beta_n} B_{\alpha_1\dots\alpha_n}^{\beta_1\dots\beta_n}(x), \quad (87)$$

so that the spinor functions B ‘correct’ the behaviour of the light quarks and soft gluons to remind them that they are in the presence of the heavy quark, leaving the heavy quark label α ‘free’. This choice corresponds to the physical picture presented in the Introduction that in the heavy mass limit the heavy quark spin is decoupled. These functions should be considered to contain the nonperturbative information (both in the coupling and in the mass), that is needed for a bound state to emerge. One should expect that, while these are certainly functions of the heavy mass, there may be no sense in which they can be expanded in a $1/m_Q$ series. For example, such a B might have a dependence of the form

$$\sqrt{m_Q/\mu^3} \exp(-m_Q(v_Q \cdot k)^2/\mu^3), \quad (88)$$

which is a sum of derivatives of a delta function in a $1/m_Q$ expansion and which, consequently, may yield quite divergent integrals term by term. This is reminiscent of the non-analytic behaviour one expects in terms of the strong coupling in the confining region in QCD.

If one accepts, for the moment, the need to introduce the B -functions into the definition of the wave function, then it is apparent that the discussion on the normalization of the wave function needs to take this into account. One would have to insert B in the appropriate places in (134), (135), (136) and (138).

With the introduction of the B factor, the form of the matrix element (86) does not change but now $M(y, x)$ is defined as

$$M(y, x)_\beta{}^\alpha = \text{Tr} B_\beta^\delta(y) \psi_{q\delta}(y) \bar{\psi}_q^\gamma(x) \bar{B}_\gamma^\alpha W \begin{bmatrix} x & 0 & 0 & y \\ v_c & v_c & v_b & v_b \end{bmatrix}. \quad (89)$$

As $M(x, y)$ is in any case unknown, this ‘improvement’ will not effect any of our subsequent discussion on the number of form factors. One diagonal contribution to the function B is the correction coming from the matching of HQET to QCD at one loop level for example (see Section 6).

3.3 Bargmann-Wigner Wave Functions and Interpolating Fields

In order to extract useful information from the overlap integral for the transition (80) we need to determine an appropriate set of projectors χ that appear in the interpolating fields. It is well known [34] that a wide variety of interpolating fields can be used to represent a particular bound state within a reduction formula. We shall show in this subsection that the Bargmann-Wigner wave functions are the natural choice for these projectors in the heavy quark limit [35].

As a first example consider a heavy vector meson. An interpolating field $\phi_\mu(x)$ for a vector particle, with momentum P , should satisfy the condition

$$\langle 0 | \phi_\mu(x) | P \rangle = \epsilon_\mu e^{-iP \cdot x}, \quad (90)$$

where the polarisation vector ϵ_μ satisfies the transversality condition $v \cdot \epsilon = 0$ and $\epsilon^{*\mu} \epsilon_\mu = -1$.

Thus, we can take as interpolating fields, for example, either

$$\phi_\mu^1(x) = N_1 T \bar{\psi}_q(x) \gamma_\mu^\perp \psi_Q(x), \quad (91)$$

or

$$\phi_\mu^2(x) = -N_2 \frac{i\partial_\nu}{M_Q} T \bar{\psi}_q(x) \gamma^\nu \gamma_\mu^\perp \psi_Q(x). \quad (92)$$

The normalisation constants

$$N_1 = -\frac{1}{\langle 0 | T \bar{\psi}_q(0) \not{v} \psi_Q(0) | P \rangle}, \quad (93)$$

and

$$N_2 = \frac{1}{\langle 0 | T \bar{\psi}_q(0) \not{v} \not{v} \psi_Q(0) | P \rangle} \quad (94)$$

follow from the normalisation of the polarisation vector.

Now one can use the equations of motion

$$(i\mathcal{D} - m)\psi = 0 \quad (95)$$

for the quark fields to obtain

$$\begin{aligned} & -\frac{i\partial_\nu}{M_Q} T\bar{\psi}_q(x) \gamma^\nu \gamma_\mu^\perp \psi_Q(x) \\ &= \frac{(m_Q + m_q)}{M_Q} [T\bar{\psi}_q(x) \gamma_\mu^\perp \psi_Q(x) - 2iT\bar{\psi}_q(x) \frac{\vec{D}_\mu^\perp}{m_Q + m_q} \psi_Q(x)]. \end{aligned} \quad (96)$$

We see that the two interpolating fields differ by a term which can be neglected in the heavy quark limit. Thus in the heavy quark limit, with $m_Q + m_q = M_Q$, the two interpolating fields, ϕ_μ^1 and ϕ_μ^2 , are equal along with the important relation

$$N_1 = N_2 = N. \quad (97)$$

Note that the term neglected, D_μ^\perp/M_Q , is exactly the kind of term which was dropped in deriving the HQET. One can also look at this result in another way. $i\bar{\psi}_q(x) \frac{\vec{D}_\mu^\perp}{M_Q} \psi_Q(x)$ is in fact another possible candidate for the interpolating field for the vector meson but its overlap with the physical state becomes negligible in the heavy mass limit. Note also that $i\bar{\psi}_q(x) \frac{\vec{\partial}_\mu^\perp}{M_Q} \psi_Q(x)$ corresponds to a p-wave contribution to the vector state coming from the anti-quark part of the heavy quark. In the zeroth order HQET recall that the antiquark part of the heavy quark field is suppressed and thus the overlap of this interpolating field with the physical heavy meson state becomes very small in this limit.

The pseudoscalar case is simpler. One can show using equations of motion that the two interpolating fields

$$\phi^1(x) = P_1 T\bar{\psi}_q(x) \gamma_5 \psi_Q(x), \quad (98)$$

and

$$\phi^2(x) = -P_2 \frac{i\partial_\nu}{M_Q} T\bar{\psi}_q(x) \gamma^\nu \gamma_5 \psi_Q(x), \quad (99)$$

are equal along with the relation between the normalisation constants

$$\frac{P_1}{P_2} = \frac{m_Q + m_q}{M_Q}. \quad (100)$$

Note again that, as in the vector case, in the heavy quark limit, where $m_Q + m_q \rightarrow M_Q$, $P_1/P_2 \rightarrow 1$.

Having established relationships between interpolating fields we now address the question: what are the consequences of the freedom of choice of interpolating

fields for a matrix elements involving heavy mesons and baryons? As an example we shall consider an arbitrary matrix element involving a heavy vector meson in the initial state

$$\mathcal{M} = \langle P' | J | P \rangle, \quad (101)$$

where the state $|P\rangle$ represents an incoming heavy vector meson, with momentum P and mass M_Q and the final state is arbitrary as is the current J . Now use the reduction theorem to write this matrix element as

$$\mathcal{M} = \lim_{P^2 \rightarrow M_Q^2} (P^2 - M_Q^2) \int d^4x e^{-iP.x} \langle P' | J \phi^\dagger(x) | 0 \rangle, \quad (102)$$

The $\phi^\dagger(x)$ in (102) $\epsilon^\mu \phi_\mu^\dagger$ where ϕ_μ is given in eq. (91) or (92). Thus one can write

$$\mathcal{M} = \lim_{P^2 \rightarrow M_Q^2} (P^2 - M_Q^2) N \int d^4x e^{-iP.x} \langle P' | J \chi_\alpha^\beta (\partial_x) \bar{\psi}_Q^\alpha(x) \psi_{q\beta}(x) | 0 \rangle, \quad (103)$$

where $\chi(\partial_x)$ is either \not{e} or $\frac{i\partial_x}{M_Q} \not{e}$ and correspondingly N is either N_1 or N_2 .

The statement that one can use any one of the alternative interpolating fields means that they should give the same result in the reduction theorems. To see what this implies consider the second interpolating field, i.e $\chi(\partial_x) = \frac{i\partial_x}{M_Q} \not{e}$ in the matrix element (103). Integrating by parts we can shift the x derivative in χ onto $\exp(-iP.x)$ to obtain

$$\mathcal{M} = \lim_{P^2 \rightarrow M_Q^2} (P^2 - M_Q^2) N_2 \int d^4x e^{-iP.x} \langle P' | J(\not{e})_\alpha^\beta \bar{\psi}_Q^\alpha(x) \psi_{q\beta}(x) | 0 \rangle, \quad (104)$$

whereas with the other choice of the interpolating field we have instead

$$\mathcal{M} = \lim_{P^2 \rightarrow M^2} (P^2 - M^2) N_1 \int d^4x e^{-iP.x} \langle P' | J(\not{e})_\alpha^\beta \bar{\psi}_Q^\alpha(x) \psi_{q\beta}(x) | 0 \rangle. \quad (105)$$

Thus comparing we see that if the two expressions (104) and (105) are to be equal, then $\not{e} = N_1/N_2$ as an operator inside the matrix element, at least in the mass shell limit for the heavy meson. In general this is not much of a restriction as the N_i are functions of the velocity v . However, we have shown above that, in the heavy quark limit, the ratio of the normalisation constants is unity and independant of the velocity, leading to $\not{e} = 1$. In fact we see from (104) and (105) that this condition means that only certain components of the quark fields entering in the interpolating fields contribute to the matrix element, i.e those satisfying

$$\begin{aligned} \bar{\psi}_Q(x) \not{e} &= \bar{\psi}_Q(x) \\ \not{e} \psi_q(x) &= -\psi_q(x). \end{aligned} \quad (106)$$

in the meson mass shell limit. In other words, in the rest frame of the *meson* on the mass shell, only the quark part of the heavy quark field and surprisingly

also only the antiquark part of the light quark field contributes in the reduction formula. The same relation holds for the pseudoscalar particle. It is obvious that this physical requirement will be enforced by taking the interpolating field for the heavy vector meson, in the heavy quark limit, to be

$$\begin{aligned}\phi_\mu(x) &= \frac{1}{2}(\phi_\mu^1 + \phi_\mu^2) \\ &= \frac{1}{2}N[T\bar{\psi}_q(x)\gamma_\mu^\perp\psi_Q(x) - \frac{i\partial_\nu}{M_Q}T\bar{\psi}_q(x)\gamma^\nu\gamma_\mu^\perp\psi_Q(x)].\end{aligned}\quad (107)$$

In fact even if we start with an arbitrary linear combination of the two fields, only the sum survives in the heavy quark limit. With this choice the matrix element picks up the correct projection operator $\frac{1+\not{v}}{2}$ to enforce the above condition (106).

The above considerations turn out to be a general feature of the heavy quark limit. Given a particular interpolating field $NT\bar{\psi}_q\Gamma\psi_Q$, then one can show, using the equations of motion, that this is equal to the interpolating field $N\frac{-i\partial_\nu}{M_Q}T\bar{\psi}_q\gamma^\nu\Gamma\psi_Q$, upto terms of order $O(\vec{D}^\perp/M_Q)$. Here, Γ is some Dirac matrix, possibly with derivatives. Thus the natural choice for the projection operator $\chi(\partial)$ appearing in the interpolating fields for heavy mesons, eqs. (69) and (70), is⁸

$$\chi_\beta^\alpha(\partial) = \frac{1}{2\sqrt{M_Q}}[(1 - \frac{i\partial}{M_Q})\Gamma]_\beta^\alpha. \quad (108)$$

Here, for example, $\Gamma = \gamma_5$ or $\Gamma = \not{v}$ for the pseudoscalar and vector meson respectively. As we have noted earlier, the projection operator always appears in an integral over the space-time coordinates along with the relevant exponential containing the momentum of the physical state, corresponding to the interpolating field. Thus, by partial integration the differentiation in the projection operator can be shifted onto the exponential, so that we can always replace the $\chi(\partial)$ of eq. (108) with

$$\chi_\beta^\alpha(v) = \frac{1}{2\sqrt{M_Q}}[(1 + \not{v})\Gamma]_\beta^\alpha. \quad (109)$$

in all reduction formulae.

Although we shown that the above form of the projection operators is the natural choice for the interpolating field for a heavy meson, we of course have the freedom to choose this form also for a light meson. This provides a uniform approach for both light and heavy mesons [26]. Similar considerations apply to the heavy and light baryon interpolating fields, containing the projection operators $\chi_{\alpha\beta\gamma}$.

⁸We have here introduced an explicit $1/\sqrt{M_Q}$ factor in the projection operator to factor out the heavy mass scale. This is related to the fact that our states are normalised relativistically: $\langle P|P'\rangle = 2E(2\pi)^3\delta^3(\vec{P} - \vec{P}')$.

One now recognises that these projection operators (109) are nothing but the well-known Bargmann-Wigner wave functions. Hence, we have shown that the task of projecting out preferred particle states using products of spin half fields can be achieved in an elegant manner through the use of the so called Bargmann-Wigner wave functions.

The Bargmann-Wigner wave functions are multi-spinor wave functions of some given rank and symmetry type,

$$\chi_{\alpha_1 \dots \alpha_n}(v), \quad (110)$$

which satisfy the free Dirac equation on all the labels

$$\begin{aligned} (\not{v} - 1)_{\beta}^{\alpha_1} \chi_{\alpha_1 \dots \alpha_n}(v) &= 0, \\ \dots &\dots \dots = 0, \\ (\not{v} - 1)_{\beta}^{\alpha_n} \chi_{\alpha_1 \dots \alpha_n}(v) &= 0. \end{aligned} \quad (111)$$

The reason these wave functions describe particles of a given spin is amazingly simple [2]. In the rest frame the Dirac equation can be seen to be a projection by the operator $(1 + \gamma_0)$. This means that the spin labels take on not four independent values but two. Put another way, this reduces $SO(3, 1)$ down to one of its $SU(2)$ factors. After imposing the Dirac equation on all the labels, the multi-spinor becomes a product representation of $SU(2)$. On fixing to a given symmetry type one is actually considering an irreducible representation of $SU(2)$ of dimension $2s + 1$.

For example, take $\chi(v)$ to be totally symmetric of rank $2s$ satisfying (111). Then this is a wave function for a particle of spin s and even parity⁹. In the rest frame, γ_0 is the parity operator for the spin 1/2 Dirac spinor, tensor products of it being the parity operator for the multi-spinor.

A rank 2 symmetric bispinor, for example, describes a spin one particle as follows. The gamma matrices may be split into a symmetric class, $\gamma_{\mu}C$ and $\sigma_{\mu\nu}C$, and an antisymmetric class, C , γ_5C and $i\gamma_{\mu}\gamma_5C$. The symmetric bispinor is then a linear combination

$$\chi_{\{\alpha\beta\}} = \phi^{\mu}(\gamma_{\mu}C)_{\alpha\beta} + \frac{1}{2}\phi^{\mu\nu}(\sigma_{\mu\nu}C)_{\alpha\beta}. \quad (112)$$

The Bargmann-Wigner equations (111) impose the constraints that

$$v^{\mu}\phi_{\mu} = 0, \quad i\phi_{\mu\nu} = v_{\mu}\phi_{\nu} - v_{\nu}\phi_{\mu}, \quad (113)$$

so that, on taking ϕ^{μ} to be the polarisation tensor ϵ^{μ} the wave function takes the form

$$\chi_{\{\alpha\beta\}} = \frac{1}{2}[(1 + \not{v})\not{C}]_{\alpha\beta} = [P_{+}\not{C}]_{\alpha\beta}. \quad (114)$$

⁹Actually as defined only for S -waves. To get a handle on the P and higher waves allow the $\chi(v, k)$ to depend on the orbital momentum k .[27]

If one is interested, as we are, in using these multi-spinors to describe baryons and pseudoscalar mesons, then the above example is not enough. First, we may augment this with a rank 2 anti-symmetric bispinor $\chi_{[\alpha\beta]}(v)$ describing a spin zero particle. Expanding this out in the three possible antisymmetric matrices and imposing (111) we find that

$$\chi_{[\alpha\beta]} = \frac{1}{2}[(1 + \not{v})\gamma_5 C]_{\alpha\beta}. \quad (115)$$

Secondly, as the mesons are made out of quark anti-quark bound states the wave function has one covariant and one contravariant index. The contravariant index may be lowered with the help of the charge conjugation matrix C , so that one obtains a covariant bispinor whose symmetry or anti-symmetry may be declared unambiguously as above. We can then remove the charge conjugation matrix by multiplying by C^{-1} to obtain the projection operators for the pseudoscalar and vector mesons

$$\chi_{\alpha}{}^{\beta} = \frac{1}{2\sqrt{M_Q}}[(1 + \not{v})\gamma_5]_{\alpha}{}^{\beta} \quad (116)$$

and

$$\chi_{\alpha}{}^{\beta} = \frac{1}{2\sqrt{M_Q}}[(1 + \not{v})\not{C}]_{\alpha}{}^{\beta} \quad (117)$$

respectively, as expected.

For the baryons our interest rests on the rank three trispinor $\chi_{\alpha\beta\gamma}(v)$. After the Bargman-Wigner conditions are imposed each label effectively ranges over two values, so that this wave function has $2^3 = 8$ degrees of freedom. We wish to decompose this down to its irreducible parts $(1/2 \oplus 1/2 \oplus 3/2)$, under $SU(2)$. First set

$$\chi_{\alpha\beta\gamma} = \chi_{[\alpha\beta]\gamma} + \chi_{\{\alpha\beta\}\gamma}. \quad (118)$$

If we had not enforced the Bargman-Wigner conditions, then $\chi_{[\alpha\beta]\gamma}$ would be reducible and the totally antisymmetric part would have to be projected out, that is, we would have to also impose

$$\chi_{[\alpha\beta]\gamma} + \chi_{[\beta\gamma]\alpha} + \chi_{[\gamma\alpha]\beta} = 0. \quad (119)$$

However, it is easy to see that after imposing the Bargman-Wigner equations this tracelessness condition is identically satisfied. $\chi_{[\alpha\beta]\gamma}$ represents a spin half particle. On the other hand $\chi_{\{\alpha\beta\}\gamma}$ is reducible. To project away the totally symmetric part we impose the condition

$$\chi_{\{\alpha\beta\}\gamma} + \chi_{\{\beta\gamma\}\alpha} + \chi_{\{\gamma\alpha\}\beta} = 0. \quad (120)$$

Let us keep denoting the rank three spinor of mixed symmetry that satisfies this condition by $\chi_{\{\alpha\beta\}\gamma}$. It represents a spin 1/2 particle. We start with 6 degrees of

freedom from which we exclude the 4 coming from the totally symmetric component, leaving 2 components. The complete decomposition is then

$$\chi_{\alpha\beta\gamma} = \chi_{[\alpha\beta]\gamma} + \chi_{\{\alpha\beta\}\gamma} + \chi_{\{\alpha\beta\gamma\}}. \quad (121)$$

The reason for explaining this at length is that it allows for a rather nice description of the situation in the case of heavy baryons. Firstly, the totally symmetric component describes a spin 3/2 particle and so is suitable for the Σ^* and Ω^* baryons. Now for the heavy Λ and Ξ baryons where the two light quarks are expected to form an antisymmetric diquark $s = 0$ state. If in (121) we let γ be the index on the heavy quark leg then $\chi_{[\alpha\beta]\gamma}$ is just what we are looking for, and it has the form (following the same analysis as for the mesons), upto normalisation,

$$\chi_{[\alpha\beta]\gamma} = \frac{1}{2}[(1 + \not{v})\gamma_5 C]_{\alpha\beta} u_\gamma. \quad (122)$$

For the heavy Σ and Ω baryons the light diquarks are expected to be in a symmetric $s = 1$ state. Still with γ being the heavy quark label, the other spin 1/2 wave function $\chi_{\{\alpha\beta\}\gamma}$ in (121) has the two light quarks in an $s = 1$ combination. This then is the correct wave function for these baryons. There are two ways to get to an explicit form for this wave function. In the first method we follow the same route as we did for the mesons, namely we expand $\chi_{\{\alpha\beta\}\gamma}$ in terms of the symmetric gamma matrices in the α and β indices and impose the Bargmann-Wigner equations to find, again upto overall normalisation,

$$\chi_{\{\alpha\beta\}\gamma} = \frac{1}{2}\phi_\gamma^\mu[(1 + \not{v})\gamma_\mu C]_{\alpha\beta}, \quad (123)$$

with

$$v_\mu \phi_\gamma^\mu = 0 \quad ; \quad (1 - \not{v})_\alpha^\beta \phi_\beta^\mu = 0. \quad (124)$$

On imposing the tracelessness condition (120), and after a little algebra, eq. (2.81) becomes

$$\begin{aligned} \chi_{\{\alpha\beta\}\gamma} &= \frac{1}{4}[(1 + \not{v})\gamma^\mu \gamma_5 u]_\gamma [(1 + \not{v})\gamma_\mu C]_{\alpha\beta} \\ &= \frac{1}{2}[(\gamma^\mu + v^\mu)\gamma_5 u]_\gamma [(1 + \not{v})\gamma_\mu C]_{\alpha\beta}, \end{aligned} \quad (125)$$

which is the wave function as put forward in [8, 18]. Alternatively we notice that a trace condition of the above form is easily solved in terms of antisymmetric objects,

$$\chi_{\{\alpha\beta\}\gamma} = \phi_{[\alpha\gamma]} \phi_\beta + \phi_{[\beta\gamma]} \phi_\alpha, \quad (126)$$

which on also demanding (111) is

$$\chi_{\{\alpha\beta\}\gamma} = \frac{1}{2}[(1 + \not{v})\gamma_5 C]_{\alpha\gamma} u_\beta + \frac{1}{2}[(1 + \not{v})\gamma_5 C]_{\beta\gamma} u_\alpha. \quad (127)$$

This is the form advanced in [28]. The equivalence of the various forms of the wave functions was given in [26] in a slightly roundabout way. The derivation here shows that from the Bargmann-Wigner (and group theoretic) point of view one form is as natural as the other.

Likewise for the spin 3/2 wave function there are two natural forms. By expanding once more in the symmetric gamma matrices in the α and β labels, imposing the Bargmann-Wigner equations and then demanding total symmetry one finds, up to normalisation,

$$\chi_{\{\alpha\beta\gamma\}} = u_\gamma^\mu [(1 + \not{v}) \gamma_\mu C]_{\alpha\beta}. \quad (128)$$

and also that (124) is satisfied with u_γ^μ replacing ϕ_γ^μ .

The second (and obvious) approach is to begin with the expansion in terms of the symmetric gamma matrices in any two of the labels and then add symmetric permutations of these terms in all the labels. In this way we obtain, upto normalisation,

$$\chi_{\{\alpha\beta\gamma\}} = [(1 + \not{v}) \gamma_\mu C]_{\{\alpha\beta\}} u_\gamma^\mu + [(1 + \not{v}) \gamma_\mu C]_{\{\beta\gamma\}} u_\alpha^\mu + [(1 + \not{v}) \gamma_\mu C]_{\{\gamma\alpha\}} u_\beta^\mu. \quad (129)$$

3.4 Normalization to the lowest order

Here we expand the interpolating field ϕ_Q and the physical state Φ in terms of powers of the inverse heavy quark mass. ϕ_Q and Φ appear in the defining relation for the normalization constant N_Q , so that we are also led to an expansion of N_Q . Some equations are spelled out in higher orders of $(1/m_Q)$ to exhibit the expansion, and as we have need to refer to them at a later time, when we deal with the normalization at $O(1/m_Q)$. Using Eqs. (57) we can expand the interpolating fields ϕ_Q , Eqs. (69) - (72), as

$$\phi_Q = \phi_0 + \frac{1}{2m_Q} \phi_1 + \dots, \quad (130)$$

where ϕ_0 is the interpolating field in the lowest approximation. To the lowest part ϕ_0 we may associate a lowest order term in the mass expansion of the physical state. That is, we set

$$|\Phi\rangle = |\Phi^0\rangle + \frac{1}{2m_Q} |\Phi^1\rangle + \frac{1}{(2m_Q)^2} |\Phi^2\rangle + \dots. \quad (131)$$

From (67) we have for pseudoscalar mesons (the extension to other mesons and baryons is immediate)

$$N_Q \langle 0 | \bar{\psi}_q(0) \bar{\chi}(v) \psi_Q(0) | \Phi \rangle = 1, \quad (132)$$

so that we are led to

$$N_Q = N_Q^0 + \frac{1}{2m_Q} N_Q^1 + \dots, \quad (133)$$

where

$$N_Q^0 \langle 0 | \bar{\psi}_q(0) \bar{\chi}(v) Q(0) | \Phi^0 \rangle = 1, \quad (134)$$

and

$$\phi_0(x) = N_Q^0 \bar{\psi}_q(x) \bar{\chi}(v) Q(x), \quad (135)$$

and similarly for the higher order terms. It is important to realise that to zeroth order N_Q^0 is independant of the flavour of the heavy quark.

For a pseudoscalar meson we have shown above that we can take $\chi(v) = \frac{1}{2\sqrt{M_Q}}(1 + \gamma^0)\gamma_5$. Take now $v = (1, \underline{0})$ to get

$$\frac{N_Q^0}{\sqrt{M_Q}} \langle 0 | \bar{\psi}_q(0) \gamma_0 \gamma_5 Q(0) | \Phi^0 \rangle = 1, \quad (136)$$

where we have used the fact that upto zeroth order the heavy quark field is essentially Q_+ , i.e. it satisfies $\gamma_0 Q_+ = Q_+$ in the rest frame of the heavy meson.

We may relate the normalization back to the pseudoscalar decay constant f_P , for we have

$$\langle 0 | A_0(0) | P \rangle = f_P M_Q, \quad (137)$$

so that

$$\langle 0 | \bar{\psi}_q(0) \gamma_0 \gamma_5 \tilde{Q}(0) | P^0 \rangle = f_P^0 M_Q, \quad (138)$$

where f_P^0 is the zeroth order decay constant. Comparing with (136) we have the result

$$f_P^0 = \frac{1}{N_Q^0 \sqrt{M_Q}}. \quad (139)$$

Remembering that the zeroth order normalisation constant is independant of the flavour, we have thus rederived the Voloshin- Shifman [4] scaling law for the pseudoscalar decay constant.

4 Transitions from the Bethe-Salpeter Point of view

In order to make use of dynamical models of bound state wave functions it is of practical advantage to cast the theory into a form which yields the above transition as an overlap of wave functions for the two heavy particle states. To do this, we first recall the definition of the Bethe-Salpeter wave function, which is the covariant generalization of the Schrödinger wave function.

For simplicity let us concentrate on two-quark bound states for the derivation of the connection between current transitions and wave functions. The three-quark states are treated analogously. Consider a basis $|P, a\rangle$ for heavy mesons

consisting of eigenstates of the four-momentum operator P , with a denoting the eigenvalues of other commuting observables. The definition of the heavy-light Bethe-Salpeter wave function $\Phi^Q(P, x_1, x_2)$ that we take is

$$\begin{aligned}\Phi^Q(P, x_1, x_2)_\alpha^\beta &= \langle 0 | \psi_{Q\alpha}(x_1) G(x_1, x_2) \bar{\psi}_q^\beta(x_2) | P, a \rangle \\ \bar{\Phi}^Q(P, x_1, x_2)_\beta^\alpha &= \langle P, a | \bar{\psi}_Q^\alpha(x_1) \bar{G}(x_1, x_2) \psi_{q\beta}(x_2) | 0 \rangle\end{aligned}\quad (140)$$

where time-ordering is to be understood here and subsequently. α, β are spinor labels and Q, q denote the heavy and light flavours. $G(x_1, x_2)$ and $\bar{G}(x_1, x_2)$ are colour matrices chosen to make the B-S amplitudes gauge invariant. The canonical choice for them is the path-ordered exponential

$$P \exp[i g \int_{x_1}^{x_2} A \cdot dx]. \quad (141)$$

However we will not make an explicit choice yet. In our case we are interested in the $|P, a\rangle$ corresponding to a b -meson $|\Phi_b\rangle$ or to a c -meson $|\Phi_c\rangle$ ¹⁰.

The Bethe-Salpeter wave function is related to the two-body propagator [34]¹¹

$$\mathcal{K}_Q(x_1, x_2; x_3, x_4) = \langle 0 | \psi_Q(x_1) G(x_1, x_2) \bar{\psi}_q(x_2) \bar{\psi}_Q(x_3) \bar{G}(x_3, x_4) \psi_q(x_4) | 0 \rangle \quad (142)$$

by inserting a complete set of states between $\bar{\psi}_q$ and $\bar{\psi}_Q$. In this way we get

$$\mathcal{K}_Q(x_1, x_2; x_3, x_4) = \sum_{|P, a\rangle} \Phi^Q(P, x_1, x_2) \bar{\Phi}^Q(P, x_3, x_4), (t_1, t_2 \geq t_3, t_4), \quad (143)$$

where $\Phi^Q(P)$ and $\bar{\Phi}^Q(P)$ are the Bethe-Salpeter amplitudes defined above.

To get at the $b \rightarrow c$ transition, begin with the Green's function (c.f. (74))

$$\begin{aligned}\mathcal{G}(x_1, x_2; z; y_1, y_2) &= \langle 0 | \psi_c(x_1) G(x_1, x_2) \bar{\psi}_q(x_2) \bar{\psi}_c(z) \cdot \Gamma \cdot \psi_b(z) \bar{\psi}_b(y_1) \bar{G}(y_1, y_2) \psi_q(y_2) | 0 \rangle \\ &= \sum_{|p, a\rangle, |q, b\rangle} \langle 0 | \psi_c(x_1) G(x_1, x_2) \bar{\psi}_q(x_2) | p, a \rangle \langle p, a | \bar{\psi}_c(z) \cdot \Gamma \cdot \psi_b(z) | q, b \rangle \cdot \\ &\quad \langle q, b | \bar{\psi}_b(y_1) \bar{G}(y_1, y_2) \psi_q(y_2) | 0 \rangle,\end{aligned}\quad (144)$$

with $x_{10} \geq x_{20} \geq z_0 \geq y_{10} \geq y_{20}$ and Γ any matrix (for $(V - A)$ interactions it is $\gamma_\mu(1 - \gamma_5)$). Both summations are over complete sets of states. We wish to project out the two-particle bound states. To do this, one overlaps with the

¹⁰As all particle states that will appear will have only one given velocity, there is no chance of confusion in not including the corresponding momentum explicitly in the kets.

¹¹From now on we drop the Dirac indices. We shall reinstate them whenever necessary. For example, $\mathcal{K}_Q(x_1, x_2 : x_3, x_4)$ in (142) is a four spinor valued function $\mathcal{K}_{\rho\delta}^{\alpha\beta}$. There is no sum over Dirac indices in this equation.

appropriate Bethe-Salpeter wave functions and makes use of an orthonormality condition to obtain [34]

$$\langle \Phi_c | \bar{\psi}_c(z) \Gamma \psi_b(z) | \Phi_b \rangle = \int d^4 x_1 d^4 x_2 d^4 y_1 d^4 y_2 \bar{\Phi}_{P_c}^c(x_1, x_2) \mathcal{T}(x_1, x_2; z; y_1, y_2) \Phi_{P_b}^b(y_1, y_2), \quad (145)$$

where $\mathcal{T}(x_1, x_2; z; y_1, y_2)$ is defined by

$$\mathcal{G}(x_1, x_2; z; y_1, y_2) = \int d^4 x_3 d^4 x_4 d^4 y_3 d^4 y_4 \mathcal{K}_c(x_1, x_2; x_3, x_4) \mathcal{T}(x_3, x_4; z; y_3, y_4) \mathcal{K}_b(y_3, y_4; y_1, y_2), \quad (146)$$

so that $\mathcal{T}(x_3, x_4; z; y_3, y_4)$ is two-particle irreducible on both the (x_3, x_4) and (y_3, y_4) legs. $\mathcal{T}(x_3, x_4; z; y_3, y_4)$ is sometimes determined in lowest order perturbation theory to be

$$\mathcal{T}(x_3, x_4; y_3, y_4; z) = \delta^{(4)}(x_3 - z) \delta^{(4)}(y_3 - z) S_q^{-1}(x_4, y_4) \otimes \Gamma, \quad (147)$$

where $S_q(x_4, y_4) = \langle 0 | \psi_q(y_4) \bar{\psi}_q(x_4) | 0 \rangle$ is the usual light quark propagator. For our purposes we keep $\mathcal{T}(x_3, x_4; z; y_3, y_4)$ in the implicit form (146) for the present.

As mentioned previously, the effective theory tells us nothing about confinement, so that in particular it is inappropriate to use it in determining the Bethe-Salpeter wave functions (140). It is in the region of interaction that the effective theory may be used. That is, it is reasonable to expect that the transition kernel $\mathcal{T}(x_1, x_2; z; y_1, y_2)$ may be accurately determined within a $1/m_Q$ expansion for large m_Q and in a region where there are no hard gluons.

$\mathcal{T}(x_1, x_2; z; y_1, y_2)$ is to be calculated by using the defining equations (142, 144, 146) for the Greens functions and evaluating them within the effective field theory. Begin with the definition (144), replace the ψ_b and ψ_c by the zeroth order fields b and c and use (47) to go the uncoupled fields \tilde{b} and \tilde{c} to obtain

$$\begin{aligned} \mathcal{G}(x_1, x_2; z; y_1, y_2) &= \langle 0 | \psi_c(x_1) G(x_1, x_2) \bar{\psi}_q(x_2) \bar{\psi}_c(z) \cdot \Gamma \cdot \psi_b(z) \bar{\psi}_b(y_1) \bar{G}(y_1, y_2) \psi_q(y_2) | 0 \rangle \\ &= \langle 0 | W \left[\begin{smallmatrix} x_1 \\ v_c \end{smallmatrix} \right] \tilde{c}(x_1) G(x_1, x_2) \bar{\psi}_q(x_2) \bar{\tilde{c}}(z) \cdot \Gamma \cdot W \left[\begin{smallmatrix} z \\ v_c \end{smallmatrix} \right]^{-1} W \left[\begin{smallmatrix} z \\ v_b \end{smallmatrix} \right] \\ &\quad \tilde{b}(z) \bar{\tilde{b}}(y_1) W \left[\begin{smallmatrix} y_1 \\ v_b \end{smallmatrix} \right]^{-1} \bar{G}(y_1, y_2) \psi_q(y_2) | 0 \rangle \\ &= -\langle 0 | \tilde{c}(x_1) \bar{\tilde{c}}(z) | 0 \rangle \langle 0 | Tr \bar{\psi}_q(x_2) \Gamma \psi_q(y_2) G(x_1, x_2) W \left[\begin{smallmatrix} x_1 & z & z & y_1 \\ v_c & v_c & v_b & v_b \end{smallmatrix} \right] \bar{G}(y_1, y_2) | 0 \rangle. \\ &\quad \langle 0 | \tilde{b}(z) \bar{\tilde{b}}(y_1) | 0 \rangle, \end{aligned} \quad (148)$$

where in the last step we have used the fact that the transformed heavy quark fields, b and c , decouple from the gluons. The trace is over the colour indices.

Likewise we find for the two-body propagator, (142),

$$\begin{aligned}\mathcal{K}_b(x_1, x_2; x_3, x_4) &= \langle 0 | \psi_b(x_1) G(x_1, x_2) \bar{\psi}_q(x_2) \bar{\psi}_b(x_3) \bar{G}(x_3, x_4) \psi_q(x_4) | 0 \rangle \\ &= -\langle 0 | \tilde{b}(x_1) \tilde{\bar{b}}(x_3) | 0 \rangle \langle 0 | Tr \bar{\psi}_q(x_2) \psi_q(x_4) G(x_1, x_2) \\ &\quad W \left[\begin{smallmatrix} x_1 \\ v_b \end{smallmatrix} \right] W \left[\begin{smallmatrix} x_3 \\ v_b \end{smallmatrix} \right]^{-1} \bar{G}(x_3, x_4) | 0 \rangle, \end{aligned} \quad (149)$$

where there is, again, a trace over the colour indices. By inserting these into the expression (146), one is able to solve for $\mathcal{T}(x_1, x_2; z; y_1, y_2)$

$$\begin{aligned}\mathcal{T}(x_3, x_4; z; y_3, y_4) &= - \int d^4 s_1 d^4 s_2 \delta^4(x_3 - z) \delta^4(y_3 - z) \mathcal{S}_q^{-1}(v_c; x_4, s_1; x_1, x_3) \\ &\quad \tilde{\mathcal{G}}(x_1, s_1; z; y_1, s_2) \mathcal{S}_q^{-1}(v_b; s_2, y_4; y_3, y_1), \end{aligned} \quad (150)$$

where

$$\begin{aligned}\mathcal{S}_q(v_c; x_2, x_4; x_1, x_3) &= \langle 0 | Tr \psi_q(x_2) \bar{\psi}_q(x_4) G(x_1, x_2) W \left[\begin{smallmatrix} x_1 \\ v_c \end{smallmatrix} \right] W \left[\begin{smallmatrix} x_3 \\ v_c \end{smallmatrix} \right]^{-1} \bar{G}(x_3, x_4) | 0 \rangle, \end{aligned} \quad (151)$$

$$\begin{aligned}\mathcal{S}_q(v_b; y_4, y_2; y_3, y_1) &= \langle 0 | Tr \psi_q(y_4) \bar{\psi}_q(y_2) G(y_3, y_4) W \left[\begin{smallmatrix} y_3 \\ v_b \end{smallmatrix} \right] W \left[\begin{smallmatrix} y_1 \\ v_b \end{smallmatrix} \right]^{-1} \bar{G}(y_1, y_2) | 0 \rangle, \end{aligned} \quad (152)$$

and

$$\begin{aligned}\tilde{\mathcal{G}}(x_1, x_2; z; y_1, y_2) &= \langle 0 | Tr \bar{\psi}_q(x_2) \psi_q(y_2) G(x_1, x_2) W \left[\begin{smallmatrix} x_1 & z & z & y_1 \\ v_c & v_c & v_b & v_b \end{smallmatrix} \right] \bar{G}(y_1, y_2) | 0 \rangle \otimes \Gamma. \end{aligned} \quad (153)$$

Again in the above equations the traces are over the colour indices. The \mathcal{S}^{-1} are defined through

$$\int d^4 x_4 \mathcal{S}_q(v_c; x_2, x_4; x_1, x_3) \mathcal{S}_q^{-1}(v_c; x_4, s; x_1, x_3) = \delta^4(s - x_2) \quad (154)$$

and

$$\int d^4 y_4 \mathcal{S}_q^{-1}(v_b; s, y_4; y_3, y_1) \mathcal{S}_q(v_b; y_4, y_2; y_3, y_1) = \delta^4(s - y_2). \quad (155)$$

An important result follows from this. We see that in the heavy quark effective theory one can factorize the Green's function and so solve for \mathcal{T} , a situation that is not available in the full theory, except at the perturbative level.

4.1 The Form of the Bethe-Salpeter Wave Function

A comparison of (150) with (85) or (86) allows us to infer the Bethe-Salpeter wave function of the heavy meson to be of the form

$$\Phi_\alpha^{Q\beta}(x_1, x_2) = \chi_\alpha^\delta(v) A_\delta^\beta(x_1, x_2), \quad (156)$$

or in momentum space

$$\Phi_\alpha^{Q\beta}(P, k) = \chi_\alpha^\delta(v) A_\delta^\beta(P, k), \quad (157)$$

for some multi-spinor function A . This may be obtained directly from the LSZ representation of Φ^Q as follows. Writing the state $|P, a\rangle$ in (140) in terms of the interpolating field (70), one sees that the B-S amplitude for a heavy meson can be written as

$$\begin{aligned} & \Phi_\alpha^\beta(x_1, x_2) \\ &= \chi_\rho^\delta(v) N \lim_{P^2 \rightarrow M^2} (P^2 - M^2) \langle 0 | \psi_{Q\alpha}(x_1) G(x_1, x_2) \bar{\psi}_q^\beta(x_2) \\ & \quad \int d^4y e^{-iP.y} \bar{\psi}_Q^\rho(y) \psi_{q\delta}(y) | 0 \rangle \\ &= \chi_\rho^\delta(v) A_{\alpha\delta}^{\rho\beta}(x_1, x_2). \end{aligned} \quad (158)$$

The form (156) follows when we take the heavy quark fields, ψ_Q , to be the corresponding fields in the effective theory¹². Thus when we transform to the decoupled effective fields (c.f. 47) we get

$$\begin{aligned} & \Phi_\alpha^{Q\beta}(x_1, x_2) \\ &= \chi_\rho^\delta(v) N_Q \lim_{P^2 \rightarrow M^2} (P^2 - M^2) \langle 0 | \tilde{Q}_\alpha(x_1) W \begin{bmatrix} x_1 \\ v \end{bmatrix} G(x_1, x_2) \bar{\psi}_q^\beta(x_2) \\ & \quad \cdot \int d^4y e^{-iP.y} \tilde{Q}^\rho(y) W \begin{bmatrix} y \\ v \end{bmatrix}^{-1} \psi_{q\delta}(y) | 0 \rangle. \end{aligned} \quad (159)$$

Finally the γ in the decoupled heavy quark propagator is set to unity (i.e. δ_α^ρ) by the $\frac{1}{2}(1 + \gamma)$ projector of the χ , using the same procedure as in deriving (85), leading to the form of the B-S amplitude given in (156) and (157).

The B-S amplitude (156) (or 157) has the property that the heavy quark label α satisfies the Bargmann-Wigner equation.(See footnote 4). This single important fact leads to enormous simplification of matrix elements.

¹²It follows from the discussion in the subsection on the Bargmann-Wigner wave functions and interpolating fields that the form of the B-S amplitude (158) is valid also for light mesons, if we take Q to be a light quark. It only simplifies on going to the heavy decoupled effective fields.

4.2 Universal Form Factors and Symmetries of the Effective Theory

Returning once more to the transition (86), we can write it in the following compact form (it makes no difference at this point if we use (85) rather than (86) for the general features we are about to derive)

$$\sqrt{M_b M_c} \text{Tr} \bar{\Gamma} \left(\frac{1 + \not{v}_c}{2} \right) \Gamma_\mu \left(\frac{1 + \not{v}_b}{2} \right) \Gamma \Xi(v_c, v_b), \quad (160)$$

with

$$\Xi(v_c, v_b) = 4N_b N_c \int d^4 k d^4 q M(q, k). \quad (161)$$

Here $\Gamma(\bar{\Gamma})$ is either γ_5 or \not{v} (γ_5 or \not{v}^*) depending on whether we are dealing with a pseudoscalar or vector particle in the initial (final) state. Also, because of the heavy flavour symmetry present in the zeroth order HQET, the normalisation constants are equal upto $O(1/M)$ viz. $N_b = N_c = N$. We also note that Ξ is independent of the heavy meson masses as we have pulled out the mass scales and therefore it is independent of flavour.

The most general form for Ξ is

$$\Xi(v_b, v_c) = F_0(w) + \not{v}_c F_1(w) + \not{v}_b F_2(w) + \not{v}_b \not{v}_c F_3(w), \quad (162)$$

with $\omega = v_b \cdot v_c$. However, because of the projection operators $\frac{1 + \not{v}_c}{2}$ and $\frac{1 + \not{v}_b}{2}$, the matrices that go to make up Ξ may as well be replaced by the identity so that (160) takes the universal form

$$\sqrt{M_b M_c} \xi(w) \text{Tr} \bar{\Gamma} \left(\frac{1 + \not{v}_c}{2} \right) \Gamma_\mu \left(\frac{1 + \not{v}_b}{2} \right) \Gamma \quad (163)$$

with $\xi(w) = F_0 - F_1 - F_2 + F_3$. This is a very strong result, for it says that regardless of whether the transition is pseudoscalar to pseudoscalar, pseudoscalar to vector or vector to vector, there is only one form factor. The differences between these various possibilities for transitions rests completely in the different Γ 's and $\bar{\Gamma}$'s that appear in the trace. This is what is referred to as the spin symmetry present in the heavy quark effective theory. Thus doing the traces we have

$$\begin{aligned} \langle D(v_c) | \bar{c} \gamma_\lambda b | B(v_b) \rangle &= \sqrt{M_B M_D} \xi(w) (v_b + v_c)_\lambda \\ \langle D^*(v_c) | \bar{c} \gamma_\lambda b | B(v_b) \rangle &= \sqrt{M_B M_{D^*}} \xi(w) i \epsilon^{*\nu} v_b^\rho v_c^\sigma \epsilon_{\nu\rho\sigma\lambda} \\ \langle D^*(v_c) | \bar{c} \gamma_\lambda \gamma_5 b | B(v_b) \rangle &= \sqrt{M_B M_{D^*}} \xi(w) [(1 + w) \epsilon_\lambda^* - v_b \cdot \epsilon^* v_{c\lambda}] \\ \langle D^*(v_c) | \bar{c} \gamma_\lambda b | B^*(v_b) \rangle &= \sqrt{M_{B^*} M_{D^*}} \xi(w) \{ (v_b + v_c)_\lambda \epsilon^* \cdot \epsilon - v_c \cdot \epsilon \epsilon_\lambda^* - v_b \cdot \epsilon^* \epsilon_\lambda \} \\ \langle D^*(v_c) | \bar{c} \gamma_\lambda \gamma_5 b | B^*(v_b) \rangle &= \sqrt{M_{B^*} M_{D^*}} \xi(w) i \epsilon^{*\nu} (v_b + v_c)^\rho \epsilon^\sigma \epsilon_{\nu\rho\sigma\lambda} \end{aligned} \quad (164)$$

Though the physics of this situation is clear, i.e. spin information is at least of order $O(1/m_Q)$, let us also predict this spin symmetry result from a different more formal approach. Consider once more the transition as expressed by (63), and vary the c quark field according to the transformations (45). As the action is invariant under this change, one obtains a Ward identity

$$\langle \Phi'_c | \bar{\psi}_c \Gamma \psi_b | \Phi_b \rangle = \langle \Phi_c | \bar{\psi}_c \gamma_5 \not{M}_0 \Gamma \psi_b | \Phi_b \rangle, \quad (165)$$

where the state $\langle \Phi'_c |$ is the one that corresponds asymptotically to $\gamma_5 \not{M}_0 \chi$. On taking ϵ_μ to be the polarisation vector for a spin one particle with four velocity v , $M_0 = 1$ and χ to represent a pseudoscalar, we see that the new state is a vector complete with the standard form of the wave function! This allows us to relate the different χ factors, but the reader may be wondering about the overall normalization N_Φ . Applying the spin symmetry argument to (136), shows that the normalizations N_P , N_V are the same at $O(1)$.

In this case then, (165) relates the pseudoscalar to vector transition with the pseudoscalar to pseudoscalar transition, and the universal nature of heavy quark transitions is manifest. Also the Bethe-Salpeter wave functions themselves can be shown to have a universal multispinor factor A . For example, the vector meson Bethe-Salpeter wave function Φ_V^Q is related to that of the pseudoscalar meson Φ_P^Q via spin symmetry:

$$\begin{aligned} \langle 0 | \gamma_5 \not{Q}(x_1) W \left[\begin{smallmatrix} x_1 \\ v \end{smallmatrix} \right] G(x_1, x_2) \bar{\psi}_q(x_2) | P \rangle &= \langle 0 | \not{Q}(x_1) W \left[\begin{smallmatrix} x_1 \\ v \end{smallmatrix} \right] G(x_1, x_2) \bar{\psi}_q(x_2) | V \rangle, \\ \gamma_5 \not{Q} \Phi_P^Q &= \Phi_V^Q, \end{aligned} \quad (166)$$

where $|P\rangle$ and $|V\rangle$ denote pseudoscalar and vector meson states, respectively.

4.3 Normalization at q_{max}^2

One immediate consequence of the heavy quark effective theory is the normalization of the single form factors, $\xi(w)$, in the matrix elements (164), at q_{max}^2 or $w = 1$. This normalization is achieved by comparing with the conserved flavour charge number¹³

$$\langle \Phi_Q(P) | \int d^3x \bar{\psi}_Q(x) \gamma_0 \psi_Q(x) | \Phi_Q(P) \rangle = \mathbf{1}, \quad (167)$$

where Q denotes a single heavy flavour. This equation is exact in the sense that it is valid beyond the effective theory. Thus the zero component of the vector part of the transition $Q \rightarrow Q$ at zero recoil is absolutely normalized to any order in $(1/m_Q)$. This normalization is also valid if we take the limit that $m_Q \rightarrow \infty$, where we can replace the heavy fields $\psi_Q(x)$ by their effective fields

¹³In this equation and in the rest of this subsection, the $\mathbf{1}$ on the right hand side stands symbolically for $2E(2\pi)^3 \delta^3(0)$

$Q(x)$. Now we prove, using the flavour symmetry of the action Eq.(31), that this normalization at q_{max}^2 extends also to heavy flavour changing transitions at $O(1)$. Let the unitary matrix U of Eq. (32) be

$$U = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},$$

and the heavy quark in Eq. (167) be the charmed quark c . Thus we consider the transformation $c \rightarrow \beta b + \delta c$, $\bar{c} \rightarrow \bar{\beta} \bar{b} + \bar{\delta} \bar{c}$. From LSZ reduction we learn that we can replace also the c and \bar{c} in the states by the unitarily transformed ones to get

$$\begin{aligned} \mathbf{1} &= \langle \Phi_{\beta b + \delta c} | \int d^3x (\bar{b}(x) + \bar{\delta} \bar{c}(x)) \gamma_0 (\beta b(x) + \delta c(x)) | \Phi_{\beta b + \delta c} \rangle \\ &= (\beta \bar{\beta})^2 \langle \Phi_b | \int d^3x \bar{b}(x) \gamma_0 b(x) | \Phi_b \rangle + (\delta \bar{\delta})^2 \langle \Phi_c | \int d^3x \bar{c}(x) \gamma_0 c(x) | \Phi_c \rangle \\ &+ (\beta \bar{\beta})(\delta \bar{\delta}) \left(\langle \Phi_b | \int d^3x \bar{b}(x) \gamma_0 c(x) | \Phi_c \rangle + \langle \Phi_c | \int d^3x \bar{c}(x) \gamma_0 b(x) | \Phi_b \rangle \right). \end{aligned} \quad (168)$$

As U is unitary, $(\beta \bar{\beta}) + (\delta \bar{\delta}) = 1$. Using this and the fact that $q_{max}^2 = (M_b - M_c)^2$ is below threshold, so that the amplitude $\langle \Phi_c | \int d^3x \bar{c}(x) \gamma_0 b(x) | \Phi_b \rangle$ is strictly real, we conclude that also the zero component of the vector part of the transition $b \rightarrow c$ is absolutely normalized at q_{max}^2 to $O(1)$, that is

$$\langle \Phi_c(P) | \int d^3x \bar{c}(x) \gamma_0 b(x) | \Phi_b(P) \rangle = \mathbf{1}. \quad (169)$$

Comparing with the first of the equations (164) leads to the wellknown result, $\xi(1) = 1$, to $O(1)$

As the action (31) is invariant under the transformations (45), we have also a manifest spin symmetry of heavy quark transitions, as exhibited in (165) at $O(1)$. Using this we obtain absolute normalizations of spin related transitions. For example, the pseudoscalar to vector transition is normalized by taking $\Gamma = \not{v} \gamma_5 \gamma_0$ with $\epsilon = (0, 0, 0, 1)$ and $v = (1, \underline{0})$ in (165). Thus, we have

$$\begin{aligned} &\langle V_c(P) | \int d^3x \bar{c}(x) \gamma_3 \gamma_5 b(x) | P_b(P) \rangle \\ &= \langle P_c(P) | \int d^3x \bar{c}(x) \gamma_0 b(x) | P_b(P) \rangle = \mathbf{1} \end{aligned} \quad (170)$$

using the fact that γ_0 acts on $(1 + \gamma_0)/2$ to give $(1 + \gamma_0)/2$.

In Section 5 we will derive the remarkable fact that this normalization holds also to $O(1/m_Q)$.

5 Effective theory to higher orders

In this Section we will have a look at some of the interesting features and consequences of the effective theory at $O(1/m_Q)$. As the derivation of the higher

order form of the effective theory is no more difficult than the derivation of the first order form we will see firstly how to derive all the higher order terms, in the effective theory. The $O(1/m_Q)$ term is determined explicitly. Our aim then, in the rest of this Section, is to apply the formalism we have just derived to first order in the heavy quark mass.

5.1 Derivation of Higher Order Action

Let us write the action in terms of the fields \tilde{Q} (or Q) as

$$S_Q = S_Q^0(v) + \sum_{n=1}^{\infty} S_Q^n(v)/(2m_Q)^n \quad (171)$$

where $S_Q^0(v)$ was given in (19) and derived from the standard action in (59). In order to determine the $S_Q^n(v)$ we need to substitute (57) into (56) and keep terms up to $O(1/(2m_Q)^n)$. To simplify the formulae we define

$$\begin{aligned} \not{D}^{\perp} &= \not{D} - \not{v} \cdot D \\ &= \gamma^{\mu} D_{\mu}^{\perp}, \end{aligned} \quad (172)$$

which has the useful property that

$$\{\not{D}^{\perp}, \not{v}\} = 0. \quad (173)$$

D_{μ}^{\perp} is the covariant derivative transverse to the velocity.

The action (56) is then

$$\begin{aligned} S_Q &= \int \bar{Q} e^{i(\not{D}^{\perp})/(2m_Q)} (i\not{D} - m_Q) e^{i(\not{D}^{\perp})/(2m_Q)} Q \\ &= \int \bar{Q} e^{i(\not{D}^{\perp})/(2m_Q)} (i\not{D}^{\perp} + i\not{v} \cdot D - m_Q) e^{i(\not{D}^{\perp})/(2m_Q)} Q. \end{aligned} \quad (174)$$

On setting $S_Q^n(v_Q) = \bar{Q} \mathcal{O}_n Q$, from (174) we can find the \mathcal{O}_n to be

$$\mathcal{O}_n = \frac{2^n n}{(n+1)!} (\not{D}^{\perp})^{n+1} + \sum_{k=0}^n \frac{(\not{D}^{\perp})^k}{k!} i\not{v} \cdot D \frac{(\not{D}^{\perp})^{n-k}}{(n-k)!}. \quad (175)$$

Exercise: Check this equation.

This is rather too implicit. As an example we now show how to put \mathcal{O}_1 in a conventional form,

$$\begin{aligned} \mathcal{O}_1 &= -(\not{D}^{\perp})^2 - \{\not{D}^{\perp}, \not{v} \cdot D\} \\ &= -(\not{D}^2 - (v \cdot D)^2) \\ &= -(D^2 - (v \cdot D)^2) + \frac{g}{2} \sigma_{\mu\nu} F^{\mu\nu}, \end{aligned} \quad (176)$$

where

$$\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu] \text{ and } [D^\mu, D^\nu] = -igF^{\mu\nu}. \quad (177)$$

The action up to first order is then

$$S_Q = \int \bar{Q}(i\not{v}.D - m_Q)Q - \bar{Q}\left((D^2 - (v.D)^2) - \frac{g}{2}\sigma_{\mu\nu}F^{\mu\nu}\right)\frac{Q}{2m_Q} + \dots \quad (178)$$

Returning to the question of why does the Foldy-Wouthuysen transformation have the form that it does, we note a tell-tale feature already present at $O(1/m_Q)$. This is that there are no derivatives in \mathcal{O}_1 in the v direction. This is important for if there were such a derivative, there then would arise terms in diagrams, where this operator was inserted, that would go like $v.P = m_Q$ in the numerator thus raising this $O(1/m_Q)$ correction to $O(1)$. Quite generally the appearance of \not{D}^\perp in the transformations ensures that no derivatives in the direction of v appear at any order in the $1/m_Q$ expansion. One can see this quite easily from the equation (174). There the only term containing such a derivative is

$$e^{i(\not{D}^\perp)/(2m_Q)}i\not{v}.D e^{i(\not{D}^\perp)/(2m_Q)} = e^{i(\not{D}^\perp)/(2m_Q)}i\not{v}.D e^{-i(\not{D}^\perp)/(2m_Q)}\not{v}. \quad (179)$$

We now use the identity

$$e^B A e^{-B} = A + \frac{1}{n!} \sum_{n=1}^{\infty} \underbrace{[B, [B, \dots [B, A] \dots]]}_{n \text{ times}} \quad (180)$$

with $B = i(\not{D}^\perp)/(2m_Q)$ and $A = i\not{v}.D$. In this case $[B, A] = \frac{ig}{2m_Q}\gamma_\mu v_\nu F^{\mu\nu}$. Thus to all orders one does not get derivatives in the $v.D$ direction.

The operators \mathcal{O}_n are appropriate for the Q fields. Equivalent operators $\tilde{\mathcal{O}}_n$ may be found for the \tilde{Q} fields from

$$S_Q^n(v_Q) = \bar{Q}\mathcal{O}_nQ = \bar{\tilde{Q}}\tilde{\mathcal{O}}_n\tilde{Q}. \quad (181)$$

The required relationship is

$$\tilde{\mathcal{O}}_n(x) = W \begin{bmatrix} x \\ v \end{bmatrix}^{-1} \mathcal{O}_n(x) W \begin{bmatrix} x \\ v \end{bmatrix}, \quad (182)$$

and as the $\tilde{\mathcal{O}}_n$ only involve covariant derivatives, this relationship may also be expressed as

$$\tilde{\mathcal{O}}_n(A) = \mathcal{O}_n(A'), \quad (183)$$

where the gauge field A' is given in (49) and satisfies $v.A' = 0$.

In order to get an action which completely disentangles the high and low frequency states (i.e. Q_+ from Q_-) requires another set of transformations. We will not need that action and leave this as an exercise to the interested reader (see [23]).

5.2 Normalization of the interpolating field at $O(1/m_Q)$

From the normalization of the interpolating field (132), the expansion of N_Q (133) and of the physical state (131), and taking the action correction S_1 into account, we get for mesons

$$\begin{aligned} 1 &= \left(N_Q^0 + \frac{1}{2m_Q} N_Q^1 \right) \langle 0 | \bar{\psi}_q \chi \left(1 + i \frac{\not{D}^\perp}{2m_Q} \right) \tilde{Q} | \Phi^0 + \frac{1}{2m_Q} \Phi^1 \rangle \\ &\quad + N_Q^0 \langle 0 | \bar{\psi}_q \chi \tilde{Q} \frac{S_1}{2m_Q} | \Phi^0 \rangle. \end{aligned} \quad (184)$$

Here N_Q^1 can be read off to be

$$N_Q^1 = -(N_Q^0)^2 \left(\langle 0 | \bar{\psi}_q \chi i \not{D}^\perp \tilde{Q} + \psi_q \chi \tilde{Q} S_1 | \Phi^0 \rangle + \langle 0 | \bar{\psi}_q \chi \tilde{Q} | \Phi^1 \rangle \right). \quad (185)$$

If there are B -corrections (see Sec. 3.2), they must also be incorporated in (184).

5.3 $\frac{1}{m_Q}$ corrections to form factors

In this subsection we consider the $O(1/m_Q)$ corrections to the zeroth order meson transition form factor $\xi(w)$. Here we will establish that there are no $O(1/m_Q)$ corrections to the normalization, at zero recoil, of the form factor that we found in Section 4.2. The strategy is the following. One relates, once more, the vector part of the transition at equal velocity to the c -number charge. As this has already been normalized to 1 at $0(1)$, any corrections at $O(1/m_c)$ necessarily vanish. This, together with the spin symmetry, yields the desired result. We give two proofs of this result. The first is purely diagrammatic, while the second is rather more explicit. In the second method we explicitly calculate the $O(1/m_c)$ corrections to the mesonic matrix elements of the $b \rightarrow c$ vector and axial transition currents away from the zero recoil point and show that they vanish at zero recoil. The $O(1/m_b)$ correction can be evaluated in an analogous way.

There are three possible sources of corrections that need to be taken into account. One stems from the extra term that appears at this order in the action, namely S_Q^1 . The second correction is due to the fact that the fields ψ_c that appear in the reduction formulae are not just the lowest order fields c but rather $(1 + i \not{D}_c^\perp / (2m_c))c$. The last contribution can come from perturbative corrections to the B -functions.

From the methods of proof it will be apparent that *both* $1/m_c$ and $1/m_b$ corrections vanish at threshold. We therefore only address the $O(1/m_c)$ corrections directly.

5.3.1 Diagrammatic Proof

This proof was given in [23] and elaborated upon in [24]. Let all of the corrections be thought of as insertions into the lowest order diagram and represent these

insertions by a cross. The cross stands for either a momentum kick or a source of glue or both. For the $b \rightarrow c$ transition this is exhibited in fig. 2. Note that the correction at the vertex is only on the c -quark side. Also there should be (dressed) gluon lines running between all the quark lines¹⁴, which for ease of visualization are not drawn. The corresponding charm charge normalization diagrams that should sum to zero are exhibited in fig. 3. There are two crosses on the vertex corresponding to the corrections on either side of the vertex.

The charm charge normalization diagrams may be expressed as twice those of fig. 2 where once more at the vertex the cross is a correction on the right hand side only. This doubling up occurs because we may read the diagrams in any way we choose as the momentum flow is the same on either side and $Tr(\gamma_{\mu_1} \dots \gamma_{\mu_n}) = Tr(\gamma_{\mu_n} \dots \gamma_{\mu_1})$. But these last diagrams are just those of the zero component of the vector part of the $b \rightarrow c$ transition. Hence that component of the transition must vanish. To show that the axial component also vanishes, one invokes the fact that the spin symmetry also implies a normalization condition (see Section 2.7).

For this proof to work, we need to make the assumption that there is a sensible $1/m_Q$ expansion of the B -function (see Sec. 3.2). We have no control over this term, and it would require some ‘modelling’ to ascertain its form. This lies outside the scope of these lectures and we take as a working assumption, that such an expansion exists.

5.3.2 $\frac{1}{m_c}$ correction to vector and axial transition matrix elements

In this subsection we follow the proof given by Luke [21]. Let us first look at the correction coming from the fact that, upto $O(1/m_c)$, in the reduction formulae we do not have just the field c but $(1 + i\mathcal{D}_c^\perp/(2m_c))c$. This firstly leads to a modification in the currents

$$\bar{c}\Gamma b \rightarrow \bar{c}\Gamma b - \frac{1}{2m_c} \bar{c}i \not{\mathcal{D}}_\perp \Gamma b, \quad (186)$$

where the \perp subscript is with respect to v_c . Secondly the field corrections will also enter in the interpolating fields (130). These correspond to corrections to the physical state (131). The total field corrections to the $\chi\bar{c}\bar{c}\Gamma$ (when one transforms to the free fields) portion of the LSZ reduction formula are thus

$$\frac{i}{2m_c} \left(\bar{\chi}(v_c) \not{\mathcal{D}}^\perp(x) \tilde{c}(x) \bar{\tilde{c}}(z) \Gamma - \bar{\chi}(v_c) \tilde{c}(x) \bar{\tilde{c}}(z) \not{\mathcal{D}}^\perp(z) \Gamma \right), \quad (187)$$

the first term being an interpolating field correction, the second is the correction to the current (vertex correction) already written in eq. (186). The charm propagator is made up of ψ_c and the identity matrix, while $\bar{\chi}(v_c)$ projects onto the

¹⁴In the gauge $v_b \cdot A = 0$ there are no gluons at all connecting the heavy quarks. In a general gauge these do contribute, but always in the form of Wilson lines.

$(1 + \not{y}_c)$ component. One may then move the \not{D}^\perp to the right hand side of the first term without engendering any new gamma matrix structure. So as far as the gamma matrix structure is concerned both terms are of the same type. One can do better, and establish that the wave function correction vanishes completely when one goes to the mass pole. This is not needed for our subsequent analysis; but the reader might like to keep in mind that for all intents and purposes, $|\Phi^1\rangle = 0$ in (131). Thus we will only consider the vertex correction, i.e. the correction due to the modified currents. Let us now calculate this for the $B \rightarrow D$ and $B \rightarrow D^*$ transitions. Thus we begin by considering the following matrix element

$$- \frac{1}{2m_c} \langle D(v_c) | (\bar{c}i \not{D}_\perp \Gamma_\mu b)(0) | B(v_b) \rangle, \quad (188)$$

where $\Gamma_\mu = \gamma_\mu(1 - \gamma_5)$. To evaluate this consider first the matrix element

$$\langle D(v_c) | (\bar{c}i \not{D}_\lambda \Gamma_\mu b)(0) | B(v_b) \rangle. \quad (189)$$

From our previous analysis, in Sections 3 and 4, we see that, because \not{D}_λ does not contain any gamma matrices, this matrix element can be written immediately as

$$\begin{aligned} & \sqrt{M_B M_D} Tr \gamma_5 \left(\frac{1 + \not{y}_c}{2} \right) \Gamma_\mu \left(\frac{1 + \not{y}_b}{2} \right) \gamma_5 [Av_{b\lambda} + Bv_{c\lambda} + C\gamma_\lambda] \\ &= \sqrt{M_B M_D} (v_b + v_c)_\mu (Av_{b\lambda} + Bv_{c\lambda}) \\ & \quad - C[g_{\mu\lambda}(1 - w) + v_{c\mu}v_{b\lambda} + v_{c\lambda}v_{b\mu}] - C4i\epsilon_{\mu\lambda\rho\sigma}v_c^\rho v_b^\sigma, \end{aligned} \quad (190)$$

where A, B, C are unknown functions of $w = v_b \cdot v_c$. Now multiply by v_c^λ and use the equation of motion

$$\bar{c}i v_c \cdot \not{D} = -m_c \bar{c} \not{y}_c \quad (191)$$

to write

$$-m_c \langle D(v_c) | \bar{c} \not{y}_c \Gamma_\mu b | B(v_b) \rangle = \sqrt{M_B M_D} (Aw + B - C)(v_b + v_c)_\mu. \quad (192)$$

On the left hand side we now notice that \not{y}_c will be hit by the projector $\frac{1+\not{y}}{2}$, on going to the free (\tilde{c}) field, and therefore can be set to unity. Recall the discussion at the end of Section 3.1 where the projector has already been used to remove the \not{y}_c from the c-quark propagator. Hence we get

$$-m_c \langle D(v_c) | \bar{c} \Gamma_\mu b | B(v_b) \rangle = \sqrt{M_B M_D} (Aw + B - C)(v_b + v_c)_\mu. \quad (193)$$

Comparing with the first of equations (164), we get

$$Aw + B - C = -m_c \xi. \quad (194)$$

We also have, upto $O(1/m_Q)$,

$$\begin{aligned} \langle D(v_c) | i\partial_\lambda (\bar{c}\Gamma_\mu b) | B(v_b) \rangle &= (P_b - P_c)_\lambda |\bar{c}\Gamma_\mu b| B(v_b) \rangle \\ &= (M_B v_{b\lambda} - M_D v_{c\lambda}) (v_b + v_c)_\mu \xi(w). \end{aligned} \quad (195)$$

The left hand side of this equation can now be written as

$$\langle D(v_c) | (\bar{c}i \overset{\leftarrow}{D}_\lambda \Gamma_\mu b) | B(v_b) \rangle + \langle D(v_c) | (\bar{c}\Gamma_\mu i \vec{D} b) | B(v_b) \rangle. \quad (196)$$

The first term in the above expression we have already evaluated in eq. (190). Hence multiplying eq. (195) by v_b^λ and using the equations of motion $iv_b.Db = m_b \not{v}_b b$ and the same argument we used to derive eq. (193) we arrive at

$$A + Bw - C = (M_B - m_b) \xi(w) - M_D w \xi(w). \quad (197)$$

Comparing with eq. (194), we get

$$\begin{aligned} A(1 + w) - C &= \bar{\Lambda} \xi(w), \\ A - B &= M_D \xi(w), \end{aligned} \quad (198)$$

where $\bar{\Lambda} = (M_B - m_b) = (M_D - m_c)$.

Now let us finally look at the vertex correction

$$\begin{aligned} &-\frac{1}{2m_c} \langle D(v_c) | \bar{c}i \overset{\leftarrow}{D}_\perp \Gamma_\mu b | B(v_b) \rangle \\ &= -\frac{1}{2m_c} \langle D(v_c) | \bar{c}i \overset{\leftarrow}{D}_\lambda \gamma_\perp^\lambda \Gamma_\mu b | B(v_b) \rangle \\ &= -\frac{1}{2m_c} \sqrt{M_b M_c} Tr \gamma_5 \frac{(1 + \not{v}_c)}{2} \gamma_\perp^\lambda \Gamma_\mu \frac{(1 + \not{v}_b)}{2} \gamma_5 (Av_{b\lambda} + Bv_{c\lambda} + C\gamma_\lambda). \end{aligned} \quad (199)$$

Here the functions A, B and C are the same as in eq. (190). We see immediately that only the vector transition current contributes and the trace can be worked out to give

$$-\frac{1}{2m_c} \sqrt{M_b M_c} (v_b - v_c)_\mu (A(1 + w) - 3C), \quad (200)$$

which, on using eq. (198), becomes

$$\frac{1}{2m_c} \sqrt{M_b M_c} (v_b - v_c)_\mu (2A(1 + w) - 3\bar{\Lambda} \xi(w)). \quad (201)$$

The second kind of correction comes from the fact that we have to include the effects of the new terms which appear in the action at order $O(1/m_c)$

$$S_1 = \frac{1}{2m_c} \int Q \left((iD)^2 - (iv.D)^2 + \frac{g}{2} \sigma_{\mu\nu} F^{\mu\nu} \right). \quad (202)$$

Inserting this in the path integral leads to a correction equal to

$$\begin{aligned} & \frac{i}{2m_c} \langle D(v_c) | \int d^4x T (\bar{c}(iD_\perp)^2 c)(x)(\bar{c}\Gamma_\mu b)(0) | B(v_b) \rangle \\ & + \frac{i}{2m_c} \langle D(v_c) | \int d^4x T (\bar{c}g\sigma_{\nu\lambda}F^{\nu\lambda}c)(x)(\bar{c}\Gamma_\mu b)(0) | B(v_b) \rangle \end{aligned} \quad (203)$$

Using the reduction formulae and going to the free (tilde) fields, as in Sec. 3 we find, since $(iD_\perp)^2$ does not include γ matrices, that the first term will lead to the trace

$$\begin{aligned} & \sqrt{M_b M_c} \frac{1}{2m_c} Tr \gamma_5 \frac{(1+\not{v}_c)}{2} \Gamma_\mu \frac{(1+\not{v}_b)}{2} \gamma_5 \eta(w) \\ & = \sqrt{M_b M_c} \frac{1}{2m_c} (v_b + v_c)_\mu \eta(w) \end{aligned} \quad , \quad (204)$$

where $\eta(w)$ is an unknown function. This simple form appears because the \not{v}_c appearing in the two free c-quark propagators can be put to unity because of the projector $\bar{\chi}(v_c)$ appearing in the reduction formula.

In the second term we follow the same procedure but we see immediately that because of the $\sigma_{\nu\lambda}$, the projector $\bar{\chi}(v_c)$ arising from the reduction cannot act directly on the second of the free c-quark propagators. However we recall (22) that near the heavy quark mass pole the main contribution in the propagator comes from

$$i \frac{\frac{1}{2}(1+\not{v}_c)}{v_c \cdot p_c - m_c} . \quad (205)$$

Thus the second term leads to the following trace

$$\sqrt{M_b M_c} \frac{i}{4m_c} Tr \gamma_5 \frac{(1+\not{v}_c)}{2} \sigma_{\nu\lambda} \frac{(1+\not{v}_c)}{2} \Gamma_\mu \frac{(1+\not{v}_b)}{2} \gamma_5 G^{\nu\lambda} . \quad (206)$$

It is easy to see that the only independant form of $G^{\nu\lambda}$ is

$$G^{\nu\lambda} = D v_b^\nu \gamma^\lambda + E \gamma^\nu \gamma^\lambda , \quad (207)$$

leading to the correction

$$\sqrt{M_b M_c} \frac{1}{2m_c} [D(w-1) + 3E] (v_b + v_c)_\mu , \quad (208)$$

where D and E are unknown functions of w .

Thus the total $O(1/m_c)$ correction to the $B \rightarrow D$ decay is

$$\sqrt{M_b M_c} \frac{1}{2m_c} [(\eta + D(w-1) + 3E) (v_b + v_c)_\mu + (2A(1+w) - 3\bar{\Lambda}\xi) (v_b - v_c)_\mu] . \quad (209)$$

We can follow the same procedure for the other s-wave matrix elements to obtain finally the following set of matrix elements for flavor changing currents

upto $O(1/m_c)$.

$$\begin{aligned} & \langle D(v_c) | \bar{c} \gamma_\mu b | B(v_b) \rangle \\ &= \sqrt{M_b M_c} \left[\left(\xi(w) + \frac{1}{2m_c} \{ \eta(w) + (w-1)D(w) + 3E(w) \} \right) (v_b + v_c)_\mu \right. \\ & \quad \left. + \frac{1}{2m_c} \{ 2(1+w)A(w) - 3\bar{\Lambda}\xi(w) \} (v_b - v_c)_\mu \right], \end{aligned} \quad (210)$$

$$\begin{aligned} & \langle D^*(v_c) | \bar{c} \gamma_\mu b | B(v_b) \rangle \\ &= \sqrt{M_b M_c} \left[\left(1 + \frac{\bar{\Lambda}}{2m_c} \right) \xi(w) + \frac{1}{2m_c} (\eta(w) - E(w)) \right] i \epsilon^{*\nu} v_b^\rho v_c^\sigma \epsilon_{\nu\rho\sigma\mu}, \end{aligned} \quad (211)$$

$$\begin{aligned} & \langle D^*(v_c) | \bar{c} \gamma_\mu \gamma_5 b | B(v_b) \rangle \\ &= \sqrt{M_b M_c} \left[\epsilon_\mu^* \left\{ \xi(w) \left((1+w) - \frac{(1-w)\bar{\Lambda}}{2m_c} \right) + \frac{1}{2m_c} (1+w)(\eta(w) - E(w)) \right\} \right. \\ & \quad \left. - v_b \cdot \epsilon^* v_{c\mu} \left\{ \xi(w) \left(1 + \frac{\bar{\Lambda}}{2m_c} \right) + \frac{1}{2m_c} (\eta(w) - 2A(w) + D(w) - E(w)) \right\} \right. \\ & \quad \left. + v_b \cdot \epsilon^* v_{b\mu} \frac{1}{2m_c} (2A(w) + D(w)) \right]. \end{aligned} \quad (212)$$

(213)

Here $\epsilon^{*\nu}$ is the polarisation vector of the D^* meson.

To determine the normalisation at zero recoil it is necessary to consider the neutral current matrix elements, which are governed by the same set of form factors because of the heavy flavour symmetry. Following the same procedure as for the charged currents, we get

$$\begin{aligned} & \langle D(v_2) | \bar{c} \gamma_\mu c | D(v_1) \rangle \\ &= M_c \left[\xi(w) + \frac{2}{2m_c} (\eta(w) + (w-1)D(w) + 3E(w)) \right] (v_1 + v_2)_\mu. \end{aligned} \quad (214)$$

$$\begin{aligned} & \langle D^*(v_2) | \bar{c} \gamma_\mu c | D(v_1) \rangle \\ &= M_c \left[\left(1 + \frac{4\bar{\Lambda}}{2m_c} \right) \xi(w) \right. \\ & \quad \left. + \frac{1}{2m_c} (2\eta(w) + 2(1+w)A(w) - (1-w)D(w) - 2E(w)) \right] i \epsilon^{*\nu} v_1^\rho v_2^\sigma \epsilon_{\nu\rho\sigma\mu}. \end{aligned} \quad (215)$$

$$\begin{aligned} & \langle D^*(v_2) | \bar{c} \gamma_\mu \gamma_5 c | D(v_1) \rangle \\ &= M_c \left[\epsilon_\mu^* \left\{ \xi(w) \left((1+w) - 4(1-w) \frac{\bar{\Lambda}}{2m_c} \right) \right. \right. \\ & \quad \left. \left. + \frac{1}{2m_c} (2(1-w^2)A(w) - (1-w^2)D(w) + 2(1+w)E(w) + 2(1+w)\eta(w)) \right\} \right. \\ & \quad \left. - v_1 \cdot \epsilon^* v_{2\mu} \left\{ \xi(w) \left(1 + \frac{4\bar{\Lambda}}{2m_c} \right) + \frac{1}{2m_c} (2\eta(w) - 2(2+w)A(w) + wD(w) + 2E(w)) \right\} \right] \end{aligned}$$

$$+ v_1 \cdot \epsilon^* v_{2\mu} \frac{1}{2m_c} (2A(w) + D(w)) \Big] . \quad (216)$$

$$\begin{aligned} & \langle D^*(v_2) | \bar{c} \gamma_\mu c | D^*(v_1) \rangle \\ &= M_c \left[\left(\xi(w) + \frac{2}{2m_c} (\eta(w) - E(w)) \right) (v_1 + v_2)_\mu \epsilon_1 \cdot \epsilon_2^* \right. \\ &+ \frac{1}{2m_c} (2A(w) - D(w)) (v_1 + v_2)_\mu v_1 \cdot \epsilon_2^* v_2 \cdot \epsilon_1 \\ &+ \left\{ -\xi(w) - \frac{1}{2m_c} (2\eta(w) + 2(1+w)A(w) + (1-w)D(w) - 2E(w)) \right\} \epsilon_{2\mu}^* v_2 \cdot \epsilon_1 \\ &+ \left. \left\{ -\xi(w) - \frac{1}{2m_c} (2\eta(w) + 2(1+w)A(w) + (1-w)D(w) - 2E(w)) \right\} \epsilon_{1\mu} v_1 \cdot \epsilon_2^* \right] . \end{aligned} \quad (217)$$

Here v_1 and v_2 are the velocities of the incoming and outgoing D -mesons, respectively and the ϵ_1 and ϵ_2^* are the corresponding polarisation vectors.

We observe immediately that all the $O(1/m_c)$ corrections vanish in both the flavour changing and neutral matrix elements at the zero recoil point $w = 1$, except those terms proportional to $\eta(w)$ and $E(w)$. Next we recall that the neutral vector current is normalised at $w = 1$ since at this point, $v_1 = v_2 = v$, it is a symmetry current corresponding to the conserved c -quark number. Thus

$$\langle D(v) | \bar{c} \gamma_0 c | D(v) \rangle = 2M_c v_0 = 2M_c v_0 \left(1 + \frac{1}{2m_c} (\eta(1) + 3E(1)) \right) \quad (218)$$

and

$$\langle D^*(v) | \bar{c} \gamma_0 c | D^*(v) \rangle = 2M_c v_0 = 2M_c v_0 \left(1 + \frac{1}{2m_c} (2\eta(1) - 2E(1)) \right) . \quad (219)$$

Here we have used the fact that $\xi(1) = 1$. Thus one gets the normalisation condition

$$\eta(1) = E(1) = 0 . \quad (220)$$

Thus we arrive at the conclusion that all corrections to the zeroth order result vanish at the symmetry point $w = 1$.

One can calculate the baryon transition matrix elements upto $O(1/m_c)$, in a similar manner, using the baryon Bethe-Salpeter amplitudes developed earlier. Without going into the details we give the results, upto $O(1/m_c)$, for the transition $\Lambda_b \rightarrow \Lambda_c$ [28], [16], [17], [22], [20], [18].

$$\begin{aligned} & \langle \Lambda_c(v_c) | \bar{c} \gamma_\mu b | \Lambda_b(v_b) \rangle \\ &= \bar{u}(v_c) \gamma_\mu u(v_b) \left\{ \xi_\Lambda(w) + \frac{1}{2m_c} (\eta_\Lambda(w) + \bar{\Lambda} \xi_\Lambda(w)) \right\} \\ & - \frac{1}{2m_c} v_{b\mu} \frac{\bar{\Lambda} \xi_\Lambda}{1+w} \bar{u}(v_c) u(v_b) \end{aligned} \quad (221)$$

and

$$\begin{aligned}
\langle \Lambda_c(v_c) | \bar{c} \gamma_\mu \gamma_5 b | \Lambda_b(v_b) \rangle \\
= \bar{u}(v_c) \gamma_\mu \gamma_5 u(v_b) \left\{ \xi_\Lambda(w) + \frac{1}{2m_c} \left(\eta_\Lambda(w) - \frac{(1-w)\bar{\Lambda}\xi_\Lambda(w)}{1+w} \right) \right\} \\
- \frac{1}{m_c} \frac{\bar{\Lambda}\xi_\Lambda(w)}{1+w} v_{b\mu} \bar{u}(v_c) \gamma_5 u(v_b).
\end{aligned} \tag{222}$$

Here, as in the mesonic case, $\xi_\Lambda(w)$ and $\eta_\Lambda(w)$ are unknown functions normalised as follows, at the zero recoil point:

$$\begin{aligned}
\xi_\Lambda(1) &= 1, \\
\eta_\Lambda(1) &= 0.
\end{aligned} \tag{223}$$

6 Renormalisation and the Relationship with QCD

So far we have considered how QCD is related to the HQET theory at the classical level. That is we have fixed the tree level coefficients in (171) by relating the theories by a Foldy-Wouthuysen transformation. When it comes to renormalising the theories a discrepancy arises. One obvious difference is that the higher order operators that appear on the right hand side of (171) are non-renormalisable. Due to infinities that the insertion of such operators entails the tree level coefficients will all need to be renormalised. Indeed as the operators in the expansion have increasing mass dimension, the higher the order in $1/m_Q$ one goes to the worse the divergence becomes. In order to renormalise therefore, one will need to perform an infinite number of ‘experiments’. That problem aside, for the moment, even the HQET_0 differs from QCD once loops are taken into account.

It might come as a surprise that this is so, as we have derived the HQET from QCD, and one might have expected that knowing the renormalisation constants of QCD would be enough. The differences arise because the transformations used do not respect the regularisation that is needed to define QCD in the first place. To see how this situation comes about consider the integral

$$\int d^4 p \frac{1}{(p^2 - m_Q^2)^2} \tag{224}$$

which, when regularised with dimensional regularisation, apart from a pole has a logarithmic behaviour $\ln m_Q/\mu$. Now let us perform the integral by expanding about the mass shell $p = m_Q v + k$,

$$\int d^4 k \frac{1}{4m_Q^2 (v \cdot k)^2} \left(1 - \frac{k^2}{m_Q v \cdot k} + \dots \right) = 0. \tag{225}$$

(We have used the fact that $\int d^n k k^m = 0$ in dimensional regularisation). The discrepancy lies in the fact that, while at tree level we can keep the k (and gluonic momenta) bounded by m_Q , we are unable to do so in loops. Similar manipulations show that even for convergent integrals one gets a mismatch. Notice that the type of terms that appear in the series expansion are of the form of the higher order operators that appear in the HQET.

Is this the death knell for the HQET? No not at all. This is a common situation in effective field theories. The naive effective field theory will correctly reproduce the physics at some scale (in our case the long-distance physics, $k \ll m_Q$) but will require corrections to reproduce the physics at other scales (here the short distance physics or for typical momenta $k \geq m_Q$). Indeed one can give a systematic treatment of the short-distance corrections. As the effective theory is QCD in disguise, the ‘experiments’ that are needed to fix the renormalisation constants in front of every term on the right hand side of (171) are simply a comparison of the effective theory and QCD up to the given order at some scale where we will demand equality between the two theories. We will come to terms with what this means presently, but first we wish to establish, for external momenta $k \ll m_Q$, that at leading order QCD and HQET_0 are indeed simply related.

6.1 Renormalisability of the HQET_0

The lowest order effective theory, HQET_0 , has a power counting renormalisable action (essentially as the counting is the same as for QCD). In principle we can stop here and conclude the theory is renormalisable. There is a quicker way to arrive at this, namely that, in the gauge $v \cdot A = 0$, the heavy quark decouples altogether from the rest of the theory. The heavy quark is free, while the rest of the theory is renormalisable in this gauge.

If the reader does not particularly like this gauge, then he or she can work in a covariant gauge, again with free heavy quarks but with the inclusion of the Wilson lines (see section 2.5). Wilson lines are basic as observables in QCD and better be renormalisable. Indeed this was postulated for such operators of smooth loops by Polyakov [39] and proved shortly after by [40] and [41]. A review of these matters may be found in [42].

6.1.1 Wavefunction Renormalisation of the heavy quark

Let us calculate the wavefunction renormalisation in the Feynman gauge, as we will have need of it later on. From now on quantities with asterix superscripts are defined in HQET_0 . The same objects without the asterix are the equivalents in QCD. We wish to evaluate (with $p = mv + k$)

$$-\frac{(1 + \gamma)}{2} g^{*2} \mu^\epsilon T^a T^a \int \frac{d^n q}{(2\pi)^n} v_\mu \frac{1}{\gamma' v \cdot (p + q) - m_Q} v^\mu \frac{1}{q^2}$$

$$= -\frac{(1+\gamma)}{2} \frac{4}{3} g^{*2} \mu^\epsilon \int \frac{d^n q}{(2\pi)^n} \frac{1}{v.(k+q)q^2}. \quad (226)$$

One now makes use of the identity,

$$\frac{1}{a^n b} = \int_0^\infty d\alpha \frac{\alpha^{n-1}}{(a\alpha + b)^{n+1}} \quad (227)$$

to rewrite the self energy as

$$\begin{aligned} & -\frac{(1+\gamma)}{2} \frac{4}{3} g^{*2} \mu^\epsilon \int_0^\infty d\alpha \int \frac{d^n q}{(2\pi)^n} \frac{1}{(\alpha v.(q+k) + q^2)^2} \\ & = -\frac{(1+\gamma)}{2} \frac{4}{3} g^{*2} \mu^\epsilon \frac{i}{(4\pi)^{2-\epsilon/2}} \Gamma(\epsilon/2) \int_0^\infty d\alpha (\alpha^2/4 - \alpha v.k)^{-\epsilon/2}. \end{aligned} \quad (228)$$

In order to perform the last integral one scales

$$\alpha \rightarrow -4v.k\alpha \quad (229)$$

and then changes variables to

$$z = \frac{1}{1+\alpha} \quad (230)$$

to arrive at

$$\begin{aligned} & -\frac{(1+\gamma)}{2} \frac{4}{3} g^{*2} \mu^\epsilon \frac{i}{(4\pi)^{2-\epsilon/2}} \Gamma(\epsilon/2) \int_0^1 dz (1-z)^{-\epsilon/2} z^{\epsilon-2} \\ & = -\frac{(1+\gamma)}{2} \frac{4}{3} g^{*2} \mu^\epsilon \frac{i}{(4\pi)^{2-\epsilon/2}} (-4v.k)^{1-\epsilon} \Gamma(1-\epsilon/2) \Gamma(\epsilon-1). \end{aligned} \quad (231)$$

The singular part is easily extracted from this expression. It is

$$-i \frac{(1+\gamma)}{2} \frac{16}{3} \frac{g^{*2} \mu^\epsilon}{16\pi^2} v.k \frac{1}{\epsilon} \quad (232)$$

from which we see that the wave function renormalisation is

$$Z_Q = 1 + \frac{16}{3} \frac{g^{*2} \mu^\epsilon}{16\pi^2} \frac{1}{\epsilon}. \quad (233)$$

The renormalised inverse propagator is, therefore,

$$\Gamma_{(2,0)}^* = i \frac{(1+\gamma)}{2} v.k \left(1 - \frac{16}{3} \frac{g^{*2}}{16\pi^2} \ln \mu + \dots \right). \quad (234)$$

Now the anomalous dimension for the heavy quark γ_Q^* can be determined from the renormalisation group equation

$$\left(\mu \frac{\partial}{\partial \mu} + \beta(g^*) \frac{\partial}{\partial g^*} + 2\gamma_Q^* \right) \Gamma_{(2,0)}^* = 0. \quad (235)$$

The beta function is of order g^{*3} and so we can ignore it for present purposes to obtain

$$\gamma_Q^* = \frac{8}{3} \frac{g^{*2}}{16\pi^2}. \quad (236)$$

Before closing this section let us return to the observation that what we are really calculating are expectation values of Wilson loops. The heavy quark wavefunction renormalisation should be the composite operator renormalisation of the Wilson loops and consequently the anomalous dimension (236) ought to be the anomalous dimension of the Wilson loops with end points. Indeed it is as has been shown by [43]. The reader should bear this older literature in mind. Many of the calculations performed in the context of HQET₀ could be extracted from past work on Wilson loop renormalisation.

Exercise: Prove (227) and fill in the details of the calculation leading to (233).

Exercise: What of the renormalisation of the $\frac{(1-\gamma)}{2}$ component of the propagator?

6.2 Comparison of the HQET₀ and QCD

In the previous section we calculated the one loop anomalous dimension of the heavy quark in the lowest order HQET₀. On the other hand, the one loop anomalous dimension for a quark in QCD is,

$$\gamma_Q = -\frac{4}{3} \frac{g^2}{16\pi^2}. \quad (237)$$

So there is a mismatch between the QCD and the HQET₀ predictions. While renormalisability for the HQET₀ is not an issue, it is its relationship with QCD which requires some elaboration.

In the sequel, for simplicity, we will assume that the theory of interest is that of the heavy quark coupled to the glue, with no other matter present. The general situation can be dealt with along similar lines but is rather more involved.

6.2.1 Comparison of Gluonic 1PI Diagrams

One of the important features of the HQET at lowest order is that there are no heavy quark loops at all. This is easy to see. Heavy quark loops correspond to the fermionic determinant, which is gauge invariant, and therefore we can work in the gauge $v \cdot A = 0$ to conclude that the determinant is gauge field independent. Consequently there are no heavy fermion loops. An alternative way of saying this is that, in the rest frame, the heavy quark can only propagate forward in time, leading again to no loops. So how does this square with QCD?

In fact what is relevant here is a very important result due to Appelquist and Carazzone [44](see also [45]). Consider an n point function with only external

gluon legs whose momenta are of the order of k ($k \ll m_Q$). These authors tell us that, in renormalised QCD, for such diagrams, those which contain a heavy quark loop are suppressed by at least a factor of k/m_Q relative to those which do not contain such a loop. The detailed proof of this theorem for a simpler model plus a useful guide to the literature may be found in the book [46].

Convergent Diagrams

In order to get a feel for this part of the Appelquist Carazonne theorem, consider a diagram with $n \geq 5$ where all the external gluon lines are connected to the heavy fermion loop and where all sub-diagrams are convergent. One has the generic expression (we drop γ -matrices, traces and so on),

$$\int \left(\prod_{i=1}^F d^4 q_i \right) \delta \left(\sum_{i=1}^F q_i \right) \Gamma^F(q_i) \left(\prod_{i=1}^F \frac{1}{q_i^2} \right) \times \\ \int d^4 p \left(\prod_{i=1}^F \frac{1}{p - \sum_{j=1}^i q_j - m_Q} \right) \left(\prod_{i=1}^n \frac{1}{p + \sum_{j=1}^i k_j - m_Q} \right), \quad (238)$$

where $\Gamma^F(q_i)$ is the gluonic 1PI F -point function, (Fig.4).

Since the degree of divergence of Γ^F is $4 - F$, and as the subdiagrams are all to be convergent, we require $F \geq 5$ as well. Scale the integration variables in this formula as $q_i \rightarrow m_Q q_i$ and $p \rightarrow m_Q p$ and note that, as everything in sight is convergent, the scaling law of Γ is $\Gamma^F(m_Q q_i) = m_Q^{4-F} \Gamma^F(q_i)$. The scaling argument shows us that the diagram behaves as

$$c_0 m_Q^{4-n} + c_1 k m_Q^{3-n} + \dots \quad (239)$$

for some finite constants c_j . Such an n -point function without a heavy quark loop would, by dimensional analysis, go like

$$c k^{4-n} \quad (240)$$

validating the claim. The essence of the scaling argument that we have used is simply that, because of the presence of the mass in the fermionic loop, we can set the external momenta to zero without generating a divergence.

At the other extreme, consider the situation where none of the external lines land on the heavy quark loop, (Fig.5). This time we wish to get a handle on

$$\int \left(\prod_{i=1}^F d^4 q_i \right) \delta \left(\sum_{i=1}^F q_i \right) \Gamma^{n+F}(k, q_i) \left(\prod_{i=1}^F \frac{1}{q_i^2} \right) \int d^4 p \left(\prod_{i=1}^F \frac{1}{p - \sum_{j=1}^i q_j - m_Q} \right). \quad (241)$$

Suppose that $5 \leq F < n$. In this case the heavy quark loop behaves like m_Q^{4-F} as long as the loop momenta q are small. In any case contract the loop to a point and consider the reduced diagram. The reduced diagram's overall degree of divergence is $(F - n) < 0$, so that it superficially converges. All the subdiagrams converged and so we may conclude (by Weinberg's theorem) that the reduced graph converges. The shrinking of the loop to a point is the $m_Q \rightarrow \infty$ limit of the original diagram, so that we find that such graphs behave as $m_Q^{4-F} k^{F-n}$, which are once more suppressed with respect to diagrams without fermion loops. When $(F - n) \geq 0$, one estimates this integral by looking at the regions where all the integration momenta are small (relative to m_Q) which is the dominant contribution (again by application of Weinberg's theorem). One can show that such diagrams behave as m_Q^{4-n} . These are clearly suppressed.

Divergent Diagrams

We now have to address the question of power counting divergent diagrams. According to Appelquist and Carazzone, once one renormalises the basic divergent graphs then their inclusion into other diagrams is subdominant. The argument for this is that, heuristically, the subtracted diagrams (at the scale $\mu \ll m_Q$) behave as (here Λ is a cutoff)

$$[\ln \frac{\Lambda}{m_Q} + O(\frac{k}{m_Q})] - [\ln \frac{\Lambda}{m_Q} + O(\frac{\mu}{m_Q})] = O(\frac{k}{m_Q}, \frac{\mu}{m_Q}). \quad (242)$$

An example of this behaviour (in dimensional regularisation) is afforded by the one loop correction to the gluon self energy due to a heavy quark loop. This is

$$\Pi_{\mu\nu}(k) = (k_\mu k_\nu - k^2 \eta_{\mu\nu}) \Pi(k) \quad (243)$$

where

$$\Pi(k) = \frac{2}{3} \frac{g^2}{16\pi^2} \left(\frac{2}{\epsilon} - \gamma - 6 \int_0^1 dx x(1-x) \ln \left[\frac{m_Q^2 + k^2 x(1-x)}{4\pi\mu^2} \right] \right). \quad (244)$$

Once we subtract $\Pi(k^2) - \Pi(\mu^2)$ and substitute into diagrams, the contributions are bound to be convergent (this is the essence of the renormalisation programme), as the high k behaviour is tempered. On the other hand, for $k \ll m_Q$, the (subtracted) logarithm will behave like

$$\frac{k^2}{m_Q^2}, \quad (245)$$

as in (242).

Once these, and similarly subtracted gluonic three and four point functions, are considered in the above expressions (with $F \leq 4$), one obtains the estimates

$$k^{4-F} \times O\left(\frac{k}{m_Q}\right) \text{ or } O\left(\frac{\mu}{m_Q}\right), \quad (246)$$

again establishing the suppression of such heavy quark contributions.

The Relationship between QCD and HQET₀

We have thus shown that, in a particular (mass dependent) subtraction scheme, the n -point light particle 1PI diagrams $\Gamma_n(g, m_Q, m, \mu)$ in QCD and the n -point light particle 1PI diagrams $\Gamma_n^*(g^*, m^*, \mu)$ in the HQET₀ are related by the following simple relationship which holds to order $1/m_Q$ and for $\mu \ll m_Q$,

$$\Gamma_n(g, m_Q, m, \mu, k) = \Gamma_n^*(g^*, m^*, \mu, k). \quad (247)$$

(Here we have reintroduced all of the other light, relative to the heavy quark in question, particles).

The couplings which appear on the right hand side of this equation are those that would be obtained in QCD with the heavy quark omitted. One immediate and very important consequence of all of this is that the beta function of QCD at scales below the heavy quark mass goes over to the beta function of QCD without the heavy quark. This is the idea behind grand unified theories.

There is a technical aside that should be made and that is that Appelquist and Carazzone work in the Landau gauge. The reason for this is that in any other covariant gauge the gauge parameter itself needs to be renormalised. The theorem has been carried over to other covariant gauges. We have already seen that the gauge $v.A = 0$ can, in this respect, be problematic.

Cautionary Remark: It is worth emphasising that one should not blindly apply the decoupling theorem. Here is a counterexample. Consider the divergence of the axial current $\bar{Q}\gamma_5\gamma_\mu Q$ at one loop. The anomaly is mass independent and does not vanish no matter how heavy the heavy quark is.

When choosing the renormalisation prescription we do not wish to introduce any extra m_Q dependence into the renormalised couplings. It is thus good policy to employ some MS scheme for which the renormalisation constants are mass independent. We can relate the m_Q dependent scheme of Appelquist and Carazzone to an MS scheme in the following way. Let the unrenormalised, but dimensionally

regularised, n -point light particle 1PI diagrams be $\Gamma_{U,n}(g_0, m_{Q0}, m_0, \mu, \epsilon)$. There relationship to the renormalised $\Gamma_{i,n}(g_i, m_{Qi}, m_i, \mu)$, determined in two schemes $i = 1, 2$, (one might like to think of $i = 1$ as the mass dependent subtraction scheme used by Appelquist and Carazzone, while $i = 2$ is some MS scheme) is

$$\Gamma_{U,n}(g_0, m_{Q0}, m_0, \mu, \epsilon) = \mathcal{Z}_i^{n/2} \Gamma_{i,n}(g_i, m_{Qi}, m_i, \mu). \quad (248)$$

By construction, the $\Gamma_{i,n}(g_i, m_{Qi}, m_i, \mu)$ are finite and are related by the finite ratio

$$\Gamma_{2,n}(g_2, m_{Q2}, m_2, \mu) = \left(\frac{\mathcal{Z}_1}{\mathcal{Z}_2} \right)^{n/2} \Gamma_{1,n}(g_1, m_{Q1}, m_1, \mu). \quad (249)$$

Similar formulae hold for the HQET₀ n -point functions as well

$$\Gamma_{2,n}^*(g_2^*, m_2^*, \mu) = \left(\frac{\mathcal{Z}_1^*}{\mathcal{Z}_2^*} \right)^{n/2} \Gamma_{1,n}^*(g_1^*, m_1^*, \mu). \quad (250)$$

Now starting from (247) one can pass to an MS scheme on both sides to obtain (we have switched notation so that the MS scheme couplings are to be understood and we hope our laxness causes no difficulties),

$$\Gamma_n(g, m_Q, m, \mu, k) = Z(g, m_Q, \mu)^{n/2} \Gamma_n^*(g^*, m^*, \mu, k), \quad (251)$$

for a certain function $Z(g, m_Q, \mu)$ which is given in terms of the finite renormalisation constants \mathcal{Z}_i and \mathcal{Z}_i^* .

Exercise: What is the relationship between $Z(g, m_Q, \mu)$ and the renormalisation constants \mathcal{Z}_i and \mathcal{Z}_i^* ?

6.2.2 Comparison of Green Functions with 2 Heavy quark legs

One would hope that a simple relationship like (251) holds also for external heavy quark legs and indeed it is true. Here, however, we should specify that the momentum running through the heavy quark line is “almost” on shell. That is we should take that momentum to be $m_Q v_\mu + k_\mu$, where $k \ll m_Q$. This will ensure that we can pass to the HQET₀ (we need this condition even at tree level). Given the discussion above, we will take it for granted that there is no need to take into account heavy quark loops and that the renormalisation of the light part of the theory has been accomplished. Again we should distinguish the case of convergent versus divergent diagrams. Also, given that the HQET₀ is renormalisable, it is apparent that we only need to renormalise the heavy quark propagator and the heavy quark-heavy quark-gluon vertex, as we must do also in QCD. The following argument, which gives us our sought for relationship is a variant of that due to Feinberg [6] and to Grinstein [19].

Convergent Diagrams

Consider within QCD, Green functions $G_{2,n}$, with two external heavy quark lines. Denote by $G_{2,n}^*$ the same Green functions evaluated in HQET₀. To simplify life later on we take it for granted that there is a projection of $(1 + \not{v})/2$ on these both in QCD and in HQET₀. Set $n \geq 2$. The overall degree of divergence of such Green functions is $(1 - n) < 0$, so that, apart from subdiagrams which are divergent, these Green functions would be convergent. Consider any convergent diagram contributing to such a $G_{(2,n)}$.

According to [6], the integrals of such diagrams are dominated by the region where the loop momenta satisfy $q_i \leq K$, where $K \ll m_Q$ is the typical momentum flowing through any of the external gluon lines or k . If this is the case, then one can substitute all the heavy quark QCD propagators with the corresponding HQET₀ ones,

$$\frac{1}{\not{p} + \not{q} - m_Q} \rightarrow \frac{1}{\not{v} \cdot (p + q) - m_Q}. \quad (252)$$

The two diagrams will then agree up to subdominant contributions from which we may conclude that for such convergent diagrams $G_{(2,n)} = G_{(2,n)}^*$.

As an example of the argument presented in [6] consider the diagram fig. 6. We wish to determine the behaviour of

$$\frac{(1 + \not{v})}{2} \int d^4 q \gamma_\mu \frac{1}{\not{p} + \not{q} - m_Q} \gamma_\nu \frac{1}{q^2} \frac{1}{(q + k_1)^2} \frac{(1 + \not{v})}{2}. \quad (253)$$

Split the region of integration into a region where q^2 is less than K^2 and one greater than K^2 , so that symbolically we have

$$\int_{-\infty}^{\infty} d^4 q = \int_0^K d^4 q + \int_K^{\infty} d^4 q. \quad (254)$$

In the integral \int_0^K the momentum q is small relative to m_Q and may be ignored in the QCD quark propagator. Furthermore, up to subdominant terms, the QCD propagator becomes the HQET₀ propagator. So this part of the integration coincides in the two theories.

It is easy to see that the integral $\int_{sm_Q}^{\infty} d^4 q$, on the otherhand, is itself subdominant for $s \leq 1$ and $sm_Q > K$. In this region one can set $k = k_1 = 0$ in (253) and change variables to $q \rightarrow m_Q q$, to find

$$\int_{sm_Q}^{\infty} d^4 q \gamma_\mu \frac{1}{\not{v} m_Q + \not{q} - m_Q} \gamma_\nu \frac{1}{q^4} \rightarrow \frac{1}{m_Q} \int_s^{\infty} d^4 q \gamma_\mu \frac{1}{\not{v} + \not{q} - 1} \gamma_\nu \frac{1}{q^4}. \quad (255)$$

This is clearly subdominant and shows us that the integration momenta which lie somewhat above K are irrelevant.

Divergent Diagrams

What of $G_{2,n}$ for $n = 0, 1$? Focus on the one loop correction to the three point functions $\Gamma_{(2,1)}$ and $\Gamma_{(2,1)}^*$. Within dimensional regularisation the integrals are finite and so up to $O(1/m_Q)$ they agree. However, both integrals diverge as we approach four dimensions and need counterterms. Unfortunately there is no guarantee that the counterterms preserve the relationship. If we differentiate with respect to the residual momentum or the external gluon momentum, the differentiated diagrams are finite. Then the previous arguments imply that

$$\begin{aligned}\frac{\partial}{\partial k_\mu} \Gamma_{(2,1)} &= \frac{\partial}{\partial k_\mu} \Gamma_{(2,1)*} + O(k/m_Q, q/m_Q) \\ \frac{\partial}{\partial q_\mu} \Gamma_{(2,1)} &= \frac{\partial}{\partial q_\mu} \Gamma_{(2,1)*} + O(k/m_Q, q/m_Q).\end{aligned}\quad (256)$$

So, apart from counterterms, the two $\Gamma_{(2,1)}$ are in agreement. Now the counterterms are of the form $a\Gamma_{(2,1)}^0$ and $a^*\Gamma_{(2,1)}^{*0}$, where the superscript 0 indicates the tree diagram, and a and a^* are infinite constants. Now one chooses a and a^* to ensure equality (up to subdominant terms). However, such a choice is bound to be m_Q dependent. So at this point we have

$$\Gamma_{(2,1)}^R = \Gamma_{(2,1)}^{*R} \quad (257)$$

where R denotes the renormalised Green functions.

The Relationship Including Two Heavy Quark Lines

We can pass from this m_Q dependent renormalization scheme to some mass independent scheme (as we saw above) by multiplying the Green functions by some finite renormalisation constant, so that

$$\Gamma_{(2,1)}^R(p, q; \mu) = C(m_Q/\mu, g_s) Z(m_Q, \mu, g_s) \Gamma_{(2,1)}^{*R}(k, q; \mu). \quad (258)$$

After this long discourse we finally arrive at the generalisation of (251) that we sought:

$$\Gamma_{(2,n)}(g, m_Q, m, \mu, k) = C(m_Q/\mu, g) Z(g, m_Q, \mu)^{n/2} \Gamma_{(2,n)}^*(g^*, m^*, \mu, k). \quad (259)$$

The reader may be surprised that we have by-passed a discussion of the heavy quark two point function altogether. The reason for this is that, in deriving (259), we only need to know about the wave function renormalisation to get the

multiplicative factors. These are fixed between the two and three point functions by gauge invariance. Notice also that we have an equation for $C(m_Q/\mu, g)$. That is, by applying the renormalisation group both to the left and the right of (259), with $n = 0$, (and ignoring beta function contributions) we obtain

$$\begin{aligned}\mu \frac{\partial C}{\partial \mu} &= -2(\gamma_Q - \gamma_Q^*)C \\ &= \frac{g^{*2}}{4\pi^2}C.\end{aligned}\tag{260}$$

6.3 Matching

The HQET is constructed to reproduce correctly the low energy behaviour of QCD. That is, in any diagram where the external momenta are much smaller than the mass of the heavy quark one can pass from QCD to HQET, remembering to multiply by the relevant finite renormalisation constants. This means, in particular, that we can hope that the mismatch should not depend at all on very low energy physics, such as infrared singularities, physical cuts etc. However, as we pass to lower and lower energies perturbation theory becomes more and more unreliable and the equality of QCD and HQET more and more difficult to ascertain. For example there may be non-perturbative effects that ‘see’ the quark masses. These questions aside, it is apparent that there is a range of energies for which the HQET and QCD will agree to the desired accuracy in inverse powers of m_Q .

Mismatches between QCD and HQET certainly begin to set in as one approaches scales of m_Q , for in this regime the decoupling theorems are no longer applicable. In order to get agreement between QCD and the HQET one ‘matches’ them at $\mu = m_Q$. These matching conditions manifested themselves in (251), where the extra multiplicative renormalisation constants arose to take into account hard gluon exchange, missed by the decoupling theorems. One extends this to other operators. This means that if we have some operator A and calculate in QCD or in HQET_0 then the relationship is

$$\langle A(m_Q) \rangle = C_0(m_Q, \mu) \langle A_0(\mu) \rangle_0 + \frac{C_1(m_Q, \mu)}{2m_Q} \langle A_1(\mu) \rangle_0 + \dots\tag{261}$$

The notation is that on the righthand side one is working within QCD, while on the left hand side all expectation values are taken with respect to HQET_0 and the subscript j on the operators indicates that they are the operators together with the field and action corrections at order $(m_Q)^{-j}$. The Wilson co-efficients $C_j(m_Q, \mu)$ are defined by this relation.

The idea that one should match operators in this way goes back to Voloshin and Shifman [4]. See also [47]. The following example is the one originally worked out in [4].

6.3.1 Matching for $\bar{q}\Gamma Q$

Let us see that the Wilson coefficients are calculable in practice. We write

$$\langle \bar{q}\Gamma\psi_Q \rangle = C_0^\Gamma(m_Q, \mu) \langle \bar{q}\Gamma Q \rangle_0 + \dots . \quad (262)$$

The matrix Γ can be either γ_μ or $\gamma_\mu\gamma_5$. We wish to get a handle on $C_0^\Gamma(m_Q/\mu, g_s^*)$. Act on both sides of (262) with $d/d\mu$, to obtain

$$\mu \frac{dC_0^\Gamma}{d\mu} = (\gamma_\Gamma - \gamma_\Gamma^*) C_0^\Gamma, \quad (263)$$

where γ_Γ and γ_Γ^* are the anomalous dimensions of the currents $\bar{q}\Gamma\psi_Q$ and $\bar{q}\Gamma Q$ respectively.

In QCD the operators $\bar{q}\Gamma\psi_Q$, with massless quarks and $\Gamma = \gamma_\mu$, are conserved currents, and are partially conserved currents when $\Gamma = \gamma_\mu\gamma_5$. They have vanishing anomalous dimensions. The currents correspond to flavour symmetry in QCD. Adding mass terms for the quarks breaks the symmetry softly. This means that the ultraviolet properties of the theory are not changed. In loop integrals the presence of masses certainly effects the soft momentum region of integration, but they play no role in the ultraviolet region. Consequently, the currents remain finite and at most require a finite renormalisation. Thus, even if the quarks have mass, the anomalous dimension is zero. We are left with

$$\mu \frac{dC_0^\Gamma}{d\mu} = -\gamma_\Gamma^* C_0^\Gamma. \quad (264)$$

The reason that there may, nevertheless, be a non-zero anomalous dimension for the effective theory operator $\bar{q}\Gamma Q$ is that in the HQET it does not correspond to a (partially) conserved current. One can see this by noting that the heavy quark kinetic term is totally different to the usual light quark kinetic term and so there is no hope of a symmetry at this point. Consequently it is not protected from renormalisation. In practice such a renormalisation is required, as in QCD the currents exhibit a logarithmic dependance on m_Q which diverges as $m_Q \rightarrow \infty$, that is it diverges as we pass to the HQET.

The solution to (264) has been discussed in other lectures at this school and is

$$C_0^\Gamma(\mu, g^*(\mu)) = \exp \left(- \int_{\bar{g}^*(\mu_0)}^{g^*(\mu)} dg' \frac{\gamma_\Gamma^*(g')}{\beta(g')} \right) C_0^\Gamma(\mu_0, \bar{g}^*(\mu_0)), \quad (265)$$

where \bar{g} is the running coupling defined by

$$\mu' \frac{d\bar{g}(\mu')}{d\mu'} = \beta(\bar{g}(\mu')) \quad (266)$$

and with initial condition

$$\bar{g}(\mu) = g^*. \quad (267)$$

Take $\mu_0 = m_Q$ in (265) to give

$$C_0^\Gamma(m_Q/\mu, g^*(\mu)) = \exp \left(- \int_{\bar{g}^*(m_Q)}^{\bar{g}(\mu)} dg' \frac{\gamma_\Gamma^*(g')}{\beta(g')} \right) C_0^\Gamma(\mu_0, \bar{g}^*(m_Q)). \quad (268)$$

In order to determine the Wilson coefficient we need to know two things,

$$\begin{aligned} (i) \quad & \gamma_\Gamma^*(g') \quad \text{and} \\ (ii) \quad & C_0^\Gamma(1, \bar{g}(m_Q)) . \end{aligned} \quad (269)$$

Both may be calculated perturbatively provided that μ and m_Q are large enough to ensure that $\bar{g}(\mu)$ and $\bar{g}(m_Q)$ are small (i.e. $\Lambda \ll \mu \ll m_Q$).

(i) The Anomalous Dimension γ_Γ^*

We can determine the anomalous dimension of the current by demanding that the renormalisation group equation

$$\left(\frac{\partial}{\partial \mu} + \beta(g) \frac{\partial}{\partial g} + \gamma_Q^* + \gamma_q - \gamma_\Gamma^* \right) \Gamma_\Gamma^* = 0 \quad (270)$$

holds. We also know from (236) and (237) that

$$\gamma_Q^* = \frac{8}{3} \frac{g^2}{16\pi^2}, \quad \gamma_q = -\frac{4}{3} \frac{g^2}{16\pi^2} \quad (271)$$

and therefore we only need to determine the pole part of Γ_Γ^* to work out γ_Γ^* . The diagram of interest, fig. 7, corresponds to

$$-i \frac{4}{3} g^{*2} \mu^\epsilon \int \frac{d^n l}{(2\pi)^n} \frac{\not{v}(\not{l} + \not{q}) \Gamma}{l^2 (l + q)^2 v \cdot l} . \quad (272)$$

One can set $k = 0$ without engendering an infrared divergence, and as we are only interested in the pole behaviour, we do so. The q_μ in the numerator gives a convergent integral and therefore we ignore it. Furthermore, the pole part of the l_μ integral must be proportional to v_μ , and thus it is given by

$$-i \frac{4}{3} g^{*2} \mu^\epsilon \int \frac{d^n l}{(2\pi)^n} \frac{v \cdot l \Gamma}{l^2 (l + q)^2 v \cdot l} = \frac{8}{3} \frac{g^{*2}}{16\pi^2} \Gamma \cdot \frac{1}{\epsilon} . \quad (273)$$

We have then

$$\Gamma_\Gamma^* = \left(1 + \frac{8}{3} \frac{g^{*2}}{16\pi^2} \ln \mu \right) \Gamma . \quad (274)$$

Plugging this into (270) (and once more ignoring the β term) we obtain

$$\gamma_\Gamma^* = 4 \frac{g^{*2}}{16\pi^2}. \quad (275)$$

(ii) The Wilson Coefficient C_0^Γ

At tree level, QCD and the HQET agree. So we have

$$C_0^\Gamma(1, \bar{g}(m_Q)) = 1 + O(\bar{g}(m_Q)^2), \quad (276)$$

which will suffice for our needs. We only want to work at the level of leading logarithms.

Recall that

$$\beta(g) = g \left(-b_0 \frac{g^2}{16\pi^2} + b_1 \left(\frac{g^2}{16\pi^2} \right)^2 \right), \quad (277)$$

with $b_0 = 11 - \frac{2}{3}n_f$, where n_f is the number of ‘active’ quarks. Below the b -quark, but above the c -quark, threshold we have $n_f = 4$ etc.. Putting all the threads together we can determine the exponent in (265)

$$\begin{aligned} & - \int_{\bar{g}(m_Q)}^{\bar{g}(\mu)} dg' \frac{g'^2}{16\pi^2} \left[-b_0 \frac{g'^3}{16\pi^2} \right]^{-1} \\ &= \frac{4}{b_0} \int_{\bar{g}(m_Q)}^{\bar{g}(\mu)} \frac{dg'}{g'} \\ &= \frac{4}{b_0} \ln \frac{\bar{g}(\mu)}{\bar{g}(m_Q)}. \end{aligned} \quad (278)$$

We thus arrive at

$$\begin{aligned} C_0^\Gamma &= \exp \left(\frac{4}{b_0} \ln \frac{\bar{g}(\mu)}{\bar{g}(m_Q)} \right) \\ &= \left(\frac{\bar{\alpha}_s(m_Q)}{\bar{\alpha}_s(\mu)} \right)^a \end{aligned} \quad (279)$$

where

$$\bar{\alpha}_s = \frac{\bar{g}^2}{4\pi} \quad (280)$$

and

$$a = -\frac{2}{b_0}. \quad (281)$$

If one wants more precision one can go to the next to leading logarithms approximation. For this one would need to calculate γ_Γ^* to the next order.

Exercise: Determine C from (260).

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A Symmetries of the Heavy Quark Action

In this appendix we give the most general symmetries of the heavy quark action. There are 16 symmetries, 8 “vector” and 8 “axial vector” types. In order to exhibit this we need to introduce some notation. Suppose the dimension of space-time is d . Pick a γ matrix algebra for this space and let the γ matrices be $n \times n$ matrices. As a basis for the γ choose \not{v} and γ^\perp . Any $n \times n$ may be expanded in terms of a basis Γ generated by the γ . The basis we choose is such that $\Gamma = \Gamma_+ \otimes \Gamma_-$ where

$$\{\not{v}, \Gamma_+\} = 0 \tag{282}$$

and

$$[\not{v}, \Gamma_-] = 0, \tag{283}$$

depending solely on whether there are an odd or an even number of γ^\perp in the product used to define the given element of Γ . Clearly Γ_+ and Γ_- each have dimension $n^2/2$. The symmetries of $\tilde{Q}\not{v}\partial\tilde{Q}$ are

$$\begin{aligned} \delta\tilde{Q} &= (\alpha^+ \Gamma_+ + \alpha^- \Gamma_-) \tilde{Q} \\ \delta\bar{\tilde{Q}} &= \bar{\tilde{Q}} (\alpha^+ \Gamma_+ - \alpha^- \Gamma_-) \end{aligned} \tag{284}$$

Notice that all the Γ_+ symmetries are “chiral”, though, obviously, none of them is anomalous.

B Figure Captions

Figure 1 Born diagram for the electron-proton potential

Figure 2 Insertion of $O(1/m_c)$ corrections in a $b \rightarrow c$ transition

Figure 3 Insertion of $O(1/m_c)$ corrections in a $c \rightarrow c$ transition

Figure 4 Gluon n-point function where all the external gluon lines are connected to a fermion loop

Figure 5 Gluon n-point function where none of the external gluon lines land on the heavy quark loop

Figure 6 Convergent four vertex integral

Figure 7 1-loop heavy light vertex correction

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